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Glacial-Isostatic Adjustment—II. The Inverse Problem

W. R. Peltier

Department of Physics, University of Toronto, Toronto, Ontario, Canada

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Summary

The principle of correspondence is invoked to determine Laplace transform domain solutions to the surface mass loading problem for radially stratified visco-elastic (Maxwell) spheroids. These Laplace transform domain solutions are expressed in terms of visco-elastic analogues of the conventional surface load Love numbers of elasticity. These visco-elastic Love numbers may be approximately transformed to the time domain using an extremal technique. Application of this technique shows that the Love number time histories may be well approximated by the linear superposition of a discrete set of purely exponential relaxations plus a constant term. Alternatively the discrete spectrum of relaxation times involved in the synthesis of each Love number time history may be found exactly by solving the associated homogeneous problem. Such solutions determine the set of eigen-decay times associated with the normal modes of viscous gravitational relaxation of the visco-elastic planetary model. The solution of the inhomogeneous problem may be expressed in the form of a normal mode expansion. This normal mode expansion is employed as the basis for the construction of a rigorous first-order perturbation theory for the inference of the viscosity of the deep interior of the planet. A variational principle is derived which determines to first order that shift in position of a free decay pole in the relaxation spectrum which is forced by the addition of a radially-distributed perturbation of viscosity to the starting model. This determines the differential kernels required for the solution of the inverse problem. The uniqueness of the state of isostatic equilibrium for the viscously incompressible Maxwell models employed in this analysis is demonstrated and this uniqueness allows us to calculate the shift in the residue which is associated with the shift in position of a free decay pole for the inhomogeneous problem. The inhomogeneous problem is thus closed to first order. A formula is derived for the differential kernels appropriate to the inhomogeneous problem. The time domain form of these kernels may be calculated analytically. The structure of the full inverse theory is sufficiently simple that it may be employed to rigorously test the compatibility of the simple Maxwell model with the observed relaxation of the Earth's shape which accompanied deglaciation at the end of the last ice age.

1. Introduction

The inference of the viscosity of the planetary mantle from observed slow changes in the Earth's shape is a classical problem in Geophysics. It has been treated by many authors beginning with Haskell (1935, 1936, 1937), Vening Meinesz (1937), Niskanen (1948), and Heiskanen & Vening Meinesz (1958). All of these authors attempted to infer mantle viscosity by assuming that the mantle flowed like a Newtonian viscous fluid. The coefficient of viscosity was assumed to be a constant function of depth. Much of this work was concerned with the analysis of the isostatic recovery of Fennoscandia after removal of its late Wisconsin period ice sheet; a process which had begun about 20 000 years ago. However, Crittenden (1963) did the same analysis for the Lake Bonneville region and discovered that although the areal extent of the two loads differed by an order of magnitude, the relaxation times for the two regions differed only slightly, both being on the order of 5000 years. These results seemed to demand that the Newtonian flow (a transient convection) involved in the relaxation process should be confined in a layer near the surface. McConnell (1968) realized that such effective confinement could be achieved by assuming that the mantle viscosity was not uniform but increased as a function of depth. All of these conclusions and refinements of conclusions where reached, however, on the basis of the explicit assumption that the isostatic recovery of a region as large as Fennoscandia could be adequately treated in terms of a plane layered earth model. Because no solution to the full spherical self-gravitating problem existed it was not possible to test the assumption rigorously. Nevertheless, McConnell's basic conclusion that the viscosity of the mantle must increase substantially with depth in order to satisfy the relaxation time data has been widely accepted as valid.

Recently, further data has been forthcoming on the isostatic adjustment process involving a much larger areal extent of the Earth's surface. This data (Walcott 1972; Andrews 1974) is connected with the melting of the Laurentide ice sheet which disappeared over the same time interval as did the ice sheet in Fennoscandia. Data on the unloading history of this region has recently been discussed by Patterson (1972). Since this glaciation covered all of Canada and parts of the Northern United States, its melting led to the addition of a considerable water load to the ocean basins. Therefore, the region of active land emergence and submergence involved in the isostatic adjustment process extends over virtually the entire North American continent.

The existence of this new relative sea-level data set connected with a large-scale example of isostatic adjustment has encouraged several authors to construct global theories of the adjustment process. To date two such theories have been constructed; the first by Cathles (1971) and the second by Peltier (1974, hereafter referred to as Paper 1). The basic structures of these two calculations were entirely different. Both of these theories have led their authors to the same conclusion, namely that the data set from the large-scale recovery are basically incompatible with a mantle viscosity which increases substantially with depth. Cathles (1971) was led to this conclusion by comparison of the predictions of his model with relative sea-level curves and Peltier (1974) by the examination of the Green functions for the direct problem coupled with the principle of superposition. Peltier & Andrews (1976) have reinforced the conclusion further by application of the theory in Paper 1 to the generation of an extensive set of relative sea-level curves. On the basis of these global calculations both the data from the Laurentide region and from Fennoscandia appear to be in accord with a mantle viscosity that is uniform from the core–mantle boundary to the base of the lithosphere and which has a value that does not differ greatly from 10^{22} Poise (cgs).

A natural question then arises, as to why the previous calculations are not in accord with the more recent ones.

There are additional results to support the notion that the lower mantle does not have a viscosity which is much in excess of that of the upper mantle. In particular, O'Connell has shown (subject perhaps to some minor qualification) by the analysis of the non-tidal acceleration of the Earth's rotation deduced from ancient eclipse data, and by assuming that this was produced by the change in the Earth's principal moment of inertia during and after deglaciation, that the lower mantle should not have a viscosity much in excess of 10^{22} P. This idea followed Dicke's (1969) original suggestion. Again following a different tack, Goldreich & Toomre (1969) deduced on the basis of their polar wandering model that the viscosity of the lower mantle should be of the same order.

The apparent contradiction therefore, remains. Brennan (1974) has attempted to resolve this contradiction by invoking a strain rate dependent upon viscosity. This was originally suggested by Weertman (1972) as an appropriate assumption for the Earth's upper mantle, and calculation was reasonably successful. However, an alternative hypothesis must be tested before the assumption of Newtonian behaviour is discarded. For even with a load the size of the Fennoscandian ice sheet the inference of mantle viscosity at even a moderate depth in terms of a plane layered model may not be possible. The inherent lack of resolving power in McConnell's original data has been discussed by Parsons (1972) who also employed a plane layered model of the relaxation process. Assuming that McConnell's original data is accurate (and there is some question of this) we have attempted in the following analysis, to isolate the property of the spherical visco-elastic self-gravitating models, which would allow the reconciliation of McConnell's data with a mantle viscosity profile that is not a strongly increasing function of depth. This reconciliation of McConnell's data with a uniform mantle viscosity profile is a by-product of the central theme of this paper. The theme, simply stated, is to show how it is possible to construct a rigorous theory for the inference of mantle viscosity, and to thereby quantify the extent of our ignorance of this parameter.

We adopt at the outset, the implicit formulation of the problem given in Paper 1 and explicitly applied by Peltier & Andrews (1976) to the generation of relative sea-level curves for comparison with the observations. We show that within the context of this formalism it is possible to construct a theory with the required accuracy for the task we have set ourselves. The application of this formalism to the real data set will be fully described in a later article. Some of the discussion in the first two sections of the paper is basically a review of material described more fully in Paper 1, however, we require these results here and therefore, some brief repetition is necessary.

2. Theoretical basis of the forward problem

We assume that the rheological properties of the planet are amenable to description in terms of a simple linear visco-elastic (Maxwell) solid. When subject to an applied stress such a material behaves instantaneously like a Hookean elastic solid but thereafter is subject to a continuous anelastic deformation or creep. In this latter relaxation regime the material tends to behave exactly as if it were an incompressible Newtonian viscous fluid. This viscous mode of response is progressively established with ever-increasing accuracy as a function of time after the first application of the stress field. Such a simple creep mechanism is rigorously justifiable on the basis of solid-state theory only if the level of the stress field is sufficiently low. Then the material creeps by the diffusion of impurities (i.e. lattice vacancies) along and across grain boundaries (Herring 1950) and the movement of internal dislocations may be neglected. Whether or not this mechanism is appropriate to mantle material is still a matter of considerable controversy (Weertman 1970). By thoroughly examining the extent to which the Maxwell model is compatible with the observed relaxation of the Earth's shape following Pleistocene deglaciation we hope to be able to focus more sharply on this problem.

The stress–strain relation for a Maxwell solid has the following form (Malvern 1969)

$$\dot{\tau}_{ij}+(\mu/\nu)(\tau_{ij}-\tfrac{1}{3}\tau_{ii}\delta_{ij}) = 2\mu\dot{e}_{ij}+\lambda\dot{e}_{ij}\delta_{ij} \tag{1}$$

where τ_{ij} and e_{ij} are respectively the stress and strain tensors; μ and λ are the usual Lamé constants of elasticity and ν is the effective Newtonian viscosity. The dot denotes time differentiation and δ_{ij} is the usual unit diagonal tensor. Since we intend to apply the correspondence principle to analyse the mechanical behaviour of the material with stress–strain relation (1) we require the Laplace transform domain form of this relation. Performing this operation and contracting the resulting tensor relation we obtain after some manipulation the final form of the equation relating the Laplace transformed stress and strain as

$$\tilde{\tau}_{ij} = \lambda(s)\,\tilde{e}_{ii}\delta_{ij}+\mu(s)\,\tilde{e}_{ij} \tag{2}$$

where the tilde indicates Laplace transformation and where

$$\lambda(s) = \frac{\lambda s+\mu K/\nu}{s+\mu/\nu} \tag{3}$$

$$\mu(s) = \frac{\mu s}{s+\mu/\nu} \tag{4}$$

$$K_e = \lambda+\tfrac{2}{3}\mu. \tag{5}$$

The functions $\lambda(s)$ and $\mu(s)$ are compliances and K_e is the usual elastic bulk modulus. In terms of $\lambda(s)$ and $\mu(s)$ the Laplace transformed stress strain relation has exactly the same form as that for a Hookean elastic solid. The correspondence principle then assures us that if we are willing to solve an ' equivalent ' elastic problem with the stress strain relation (2) many times for different values of the Laplace transform variable s then we shall have constructed the Laplace transform of the desired solution to the problem. Some of the history of this principle and of its range of applicability is discussed in Paper 1. Cathles (1971), though he was aware of this principle, and in fact went through the formal exercise of constructing (2), made no use of it in his spherical model of the adjustment process. He rather integrated simultaneously the separate elastic and Newtonian viscous field equations. In Paper 1 it was shown that if explicit use is made of the correspondence principle then the theory assumes a much simplified form. Because of this simplicity of form, results which would otherwise prove cumbersome to obtain fall out naturally.

The appropriate ' equivalent ' elastic problem for the analysis of the surface mass loading of a visco-elastic sphere is the surface mass loading problem for an elastic sphere. This has been discussed at some length by Farrell (1972) and has been employed by him in the description of ocean tidal loading (Farrell 1972). The correct field equations for the present problem are thus the Laplace transformed and linearized (we assume that the strains produced by loading are small) equation of momentum balance and Poisson's equation. These have the form (Backus 1967)

$$\nabla.\tilde{\boldsymbol{\tau}}-\nabla(\rho g\tilde{\mathbf{u}}).\mathbf{e}_r)-\rho\nabla\tilde{\phi}+g\nabla.(\rho\tilde{\mathbf{u}})\,\mathbf{e}_r = 0 \tag{6}$$

$$\nabla^2\tilde{\phi} = -4\pi G\nabla.(\rho\tilde{\mathbf{u}}). \tag{7}$$

In equations (6) and (7) ρ is the density of the spherically stratified, hydrostatic reference state, g the radially dependent gravitational acceleration in this state, $\tilde{\mathbf{u}}$ the Laplace transformed displacement vector, $\tilde{\phi}$ the perturbation of the ambient gravitational potential (both that produced by the mass load on the surface if any and that due to variations from the reference state density), and G is Newton's gravitational constant. The full effect of self-gravitation is included. The s-dependence of solutions to these equations is contained implicitly through the form for $\tilde{\tau}$ given in equation (2). As described in Paper 1, we seek solutions to (5) and (6) for spherically symmetric earth models which have a free outer surface except perhaps at a point of loading.

In solving the inhomogeneous problem in Paper 1 we assumed that the time dependence of the surface mass load was a Dirac delta function and that its spatial dependence had the same form. On account of this particular choice for the time dependence of the load, the boundary conditions on normal and tangential stress and on the perturbation of the gravitational potential are independent of the Laplace transform variable s. Furthermore, because of the symmetry of the response to such a point load the ' equivalent ' elastic equations (5) and (6) may be reduced to the spheroidal system of equations, exactly the same system as employed in the description of the free spheroidal modes of elastic gravitational oscillation of the Earth (Backus 1967) except that in (6) the inertial forces do not appear. The spheroidal system involves three scalar variables $\tilde{u}_r$, $\tilde{u}_\theta$ and $\tilde{\phi}$ which are respectively the radial and tangential displacements and the gravitational potential perturbation. Three additional dependent variables are then introduced, these being the radial and tangential components of the stress tensor, τ_{rr} and $\tau_{r\theta}$ and a variable q related to the radial gradient of potential perturbation as is fully described in Paper 1. When each of these variables is expanded in terms of vector spherical harmonics with Legendre coefficients U_l, V_l, Φ_l, T_{rl} $T_{\theta l}$, and Q_l respectively, then the equations (5) and (6) reduce to a single matrix equation in the usual way as

$$\frac{dY}{dr} = \mathbf{A}Y \tag{8}$$

where the matrix $\mathbf{A}$ is given in Paper 1 but it is exactly the same (minus inertial terms) as that obtained in the free oscillations problem so long as the elastic Lamé constants are replaced by their compliance forms (3) and (4).

In analogy with the equivalent elastic surface loading problem, in Paper 1 we introduced dimensionless Love numbers h_l, k_l, l_l, which are now functions of the

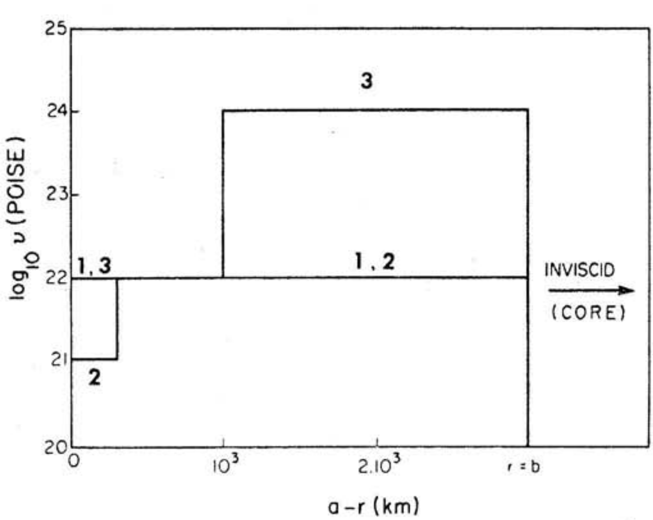

FIG. 1. Viscosity Profiles for models 1–3.

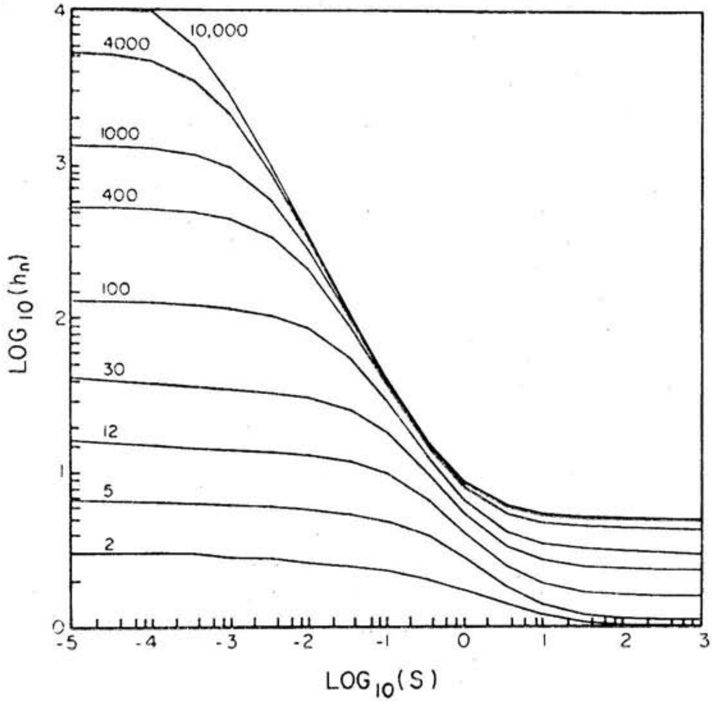

FIG. 2. Plots of s-spectra $h_n(s)$ for selected values of n (marked on figure) from viscosity model 1. Note the asymptotes for large and small s.

three variables r, l, and s. If U_l, V_l, and $\Phi_{1,l}$ arise from a force field with gravitational potential $\Phi_{2,l}(\Phi_l = \Phi_{1,l} + \Phi_{2,l})$ then these Love numbers were defined as

$$\begin{bmatrix} h_l(r, s) \\ l_l(r, s) \\ k_l(r, s) \end{bmatrix} = \frac{1}{\Phi_{2,l}(r)} \begin{bmatrix} gU_l(r, s) \\ gV_l(r, s) \\ \Phi_{1,l}(r, s) \end{bmatrix} \tag{9}$$

In Paper 1, we calculated these Love number s-spectra for several different viscosity models of the interior. Examples of Love number s-spectra viscosity models 1–3 illustrated by Fig. 1 are shown for $h_l(s)$ and $r = a$ (where a is the Earth's radius) in Figs 2, 3, and 4 respectively. The characteristics of these spectra play an important role in the formulation of the inverse problem. The large s asymptotes of these spectra are exactly the surface load Love numbers for the elastic problem which have been calculated by previous authors (e.g. Farrell 1972). The small s asymptotes are connected with the state of isostatic equilibrium as we will show. The elastic structure of all models used in this work is exactly that of the Gutenburg–Bullen model A which is described for example by Alterman, Jarosch & Pekeris (1961). The unit of time employed in all calculations is 10^3 yr and thus $s = 1$ in the above figures corresponds to a decay time of 10^3 yr. In all cases the response has been calculated for a 1 kg point mass load.

In Paper 1 we showed that these s-spectra could be transformed to the time domain (approximately) by application of an extremal technique due to Schapery (1962).

This inversion was facilitated by splitting the large s-asymptotes from each spectrum as

$$\left.\begin{aligned} h_l(s) &= h_l^V(s)+h_l^E \\ k_l(s) &= k_l^V(s)+k_l^E. \end{aligned}\right\} \tag{10}$$

After inversion we obtained time dependent viscous parts of the surface load Love numbers in the form (e.g. for $h_l^V(t)$)

$$h_l^V(t) \cong \sum_j r_j^l \exp(-ts_j^l) \tag{11}$$

where the s_j^l are a set of sampling points (pivots) of each of the Love number s-spectra and the r_j^l thus constitute a discrete approximation to a distribution function of relaxation times. These r_j^l were determined for each spectrum by the solution of a simultaneous set of linear equations which were in turn fixed by the extremal argument. Examples of the Love number time histories for the viscous part of the response are illustrated in Figs 5, 6 and 7 for the $h_l^V(t)$ and for the three viscosity models described in Fig. 1 respectively.

Given these Love number time histories, in Paper 1 we went on to construct Green functions for the several possible signatures of the process. These Green functions are obtained by summing infinite series. For example, the Green function for radial displacement in our theory has the form

$$G_R(\gamma, t) = \frac{a}{m_E} \sum_{l=0}^{\infty} [h_l^V(t)+h_l^E\,\delta(t)]\,P_l(\cos\gamma), \tag{12}$$

where the P_l are the usual Legendre polynomials and where a and m_e are respectively the Earth's radius and mass. The $\delta(t)$ dependence of the h^E part of (12) indicates that

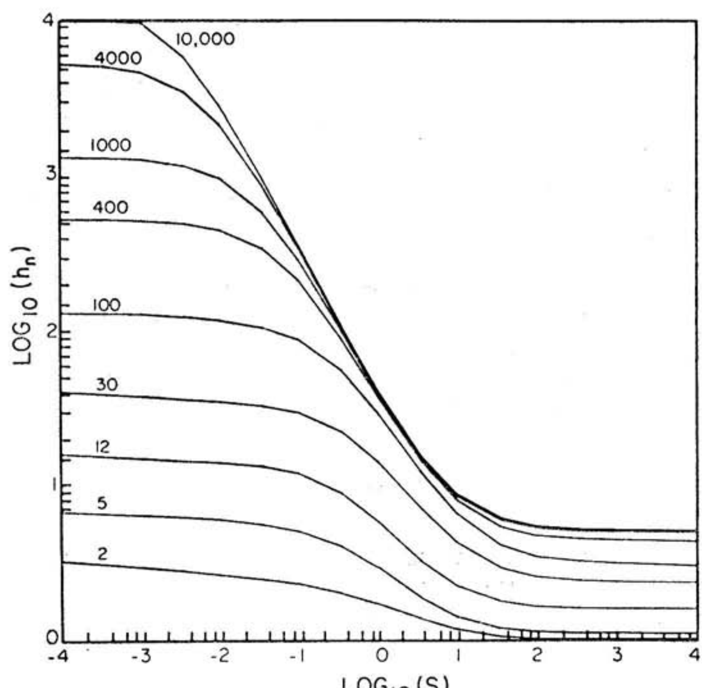

FIG. 3. Same as Fig. 2 but for viscosity model 2.

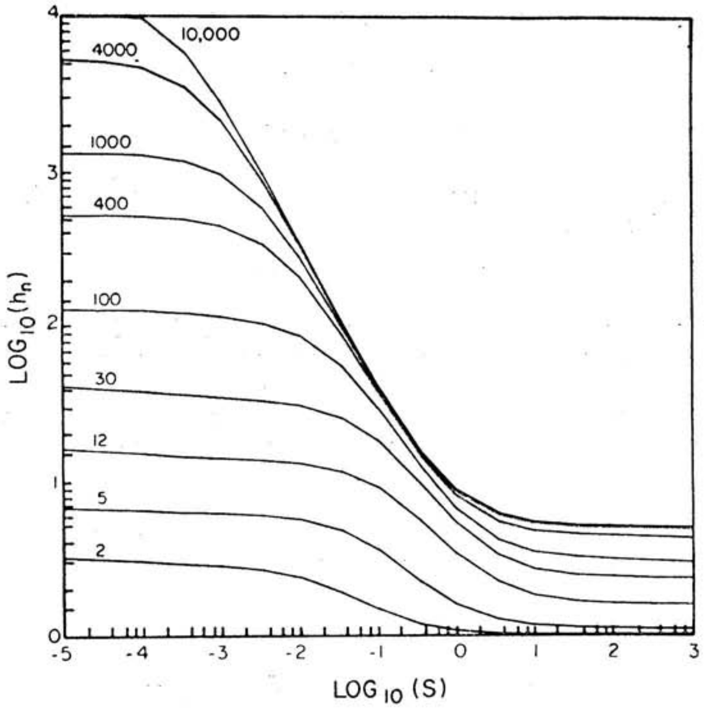

FIG. 4. Same as Fig. 2 but for viscosity model 3.

the elastic response of the planet to an applied surface load is immediate. Examples of these Green functions (viscous parts only) for radial displacement with the three viscosity models discussed previously are shown in Figs 8, 9 and 10 respectively, for viscosity models 1, 2 and 3. Given these Green functions we are in a position to write down a simple solution to the forward problem of computing the visco-elastic response of the planet to a given space and time dependent surface mass load. This solution of the forward problem is discussed more fully in Peltier & Andrews (1976).

However, if we are to make further progress with the theory we are obliged to consider critically the approximate solution of the forward problem which is implicit in the above derivation of these Green functions. In particular we might ask why the apparently crude approximation (11) seems to work so well. In Paper 1 we justified the use of the expansion (11) on the basis of a physical argument concerning the analogy with the decay spectrum of a viscous sphere (Darwin 1879). Here we treat the problem in more detail.

In the Laplace transform domain (11) has the form

$$h_l^V(s) \cong \sum_j \frac{r_j^l}{s+s_j^l}. \tag{13}$$

This approximate s-spectrum has simple poles on the negative real s-axis where $s = -s_j^l$. We note that the exact spectrum $h_l^V(s)$ is the spectrum for free decay of the special harmonic of order l in the decomposition of the non-dimensional radial displacement because the time dependence of the surface mass load has been assumed to be a Dirac function. For all $t > 0$ the surface is unloaded.

We may well ask whether the exact s-spectrum for $h_l^V(s)$ the same form as (13). This could be established directly by showing that $h_l^V(s)$ has a discrete set of infinities

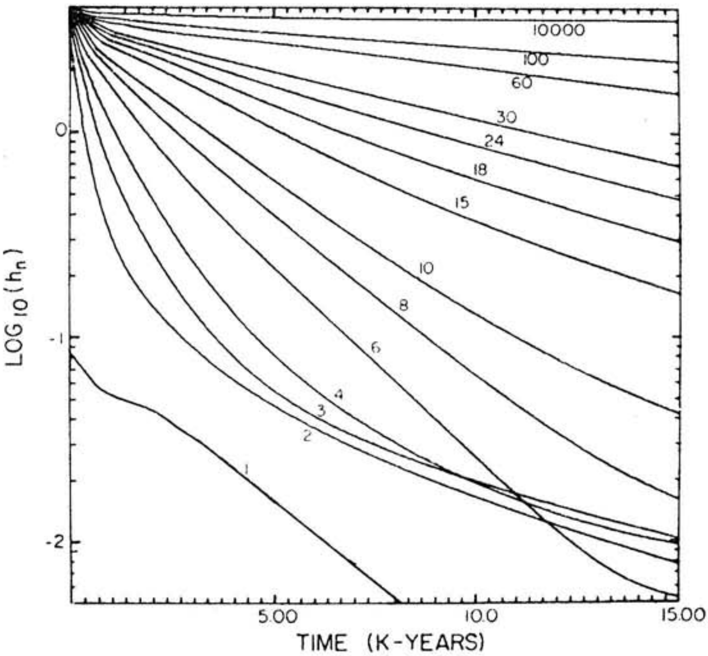

FIG. 5. Plots of time histories $h_n(t)$ for selected values of n. from viscosity model 1. Note the non-exponential decay histories for small n.

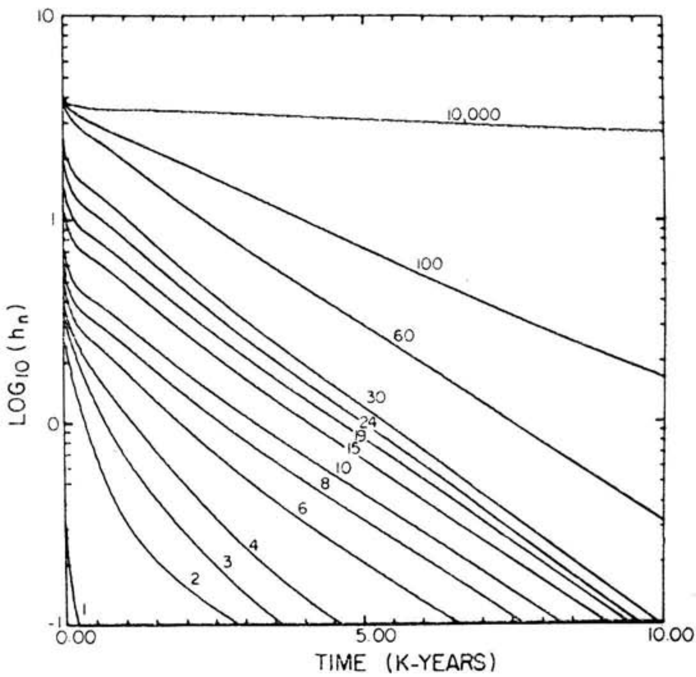

FIG. 6. Same as Fig. 5 but for viscosity model 2. Note decreased relaxation times for large n Love numbers.

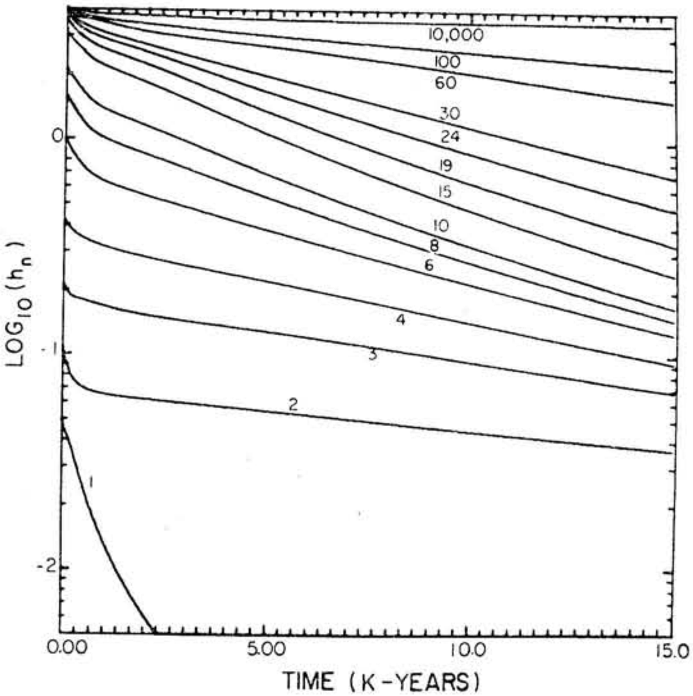

FIG. 7. Same as Fig. 6 but for viscosity model 3. Note increased relaxation times for large n Love numbers.

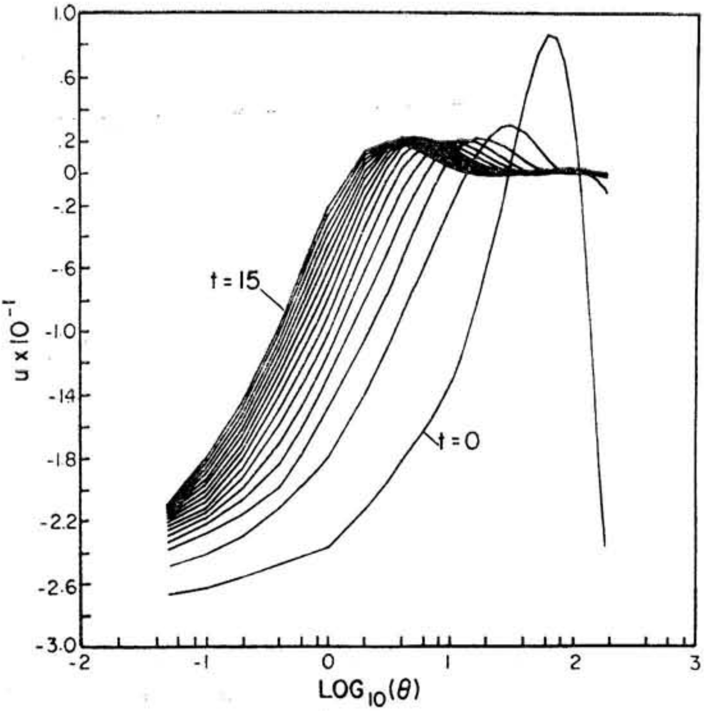

FIG. 8. Viscous part of the impulse response Green function for model 1. Note the inward migrating peripheral bulge at intermediate ϕ values.

on the negative real s-axis. However, we choose to proceed in a more conventional way. Consider the homogeneous boundary value problem associated with the simultaneous set of o.d.e.'s (8) in the Laplace transform domain. The boundary conditions for this homogeneous problem are

$$\left.\begin{aligned} T_{rl} &= 0 \\ T_{\theta l} &= 0 \\ \Phi_l &= 0 \end{aligned}\right\} \tag{14}$$

As described in Paper 1 the three linearly independent sets of starting solutions to (8) are propagated to the surface $r = a$ by numerical integration. For arbitrary s it will not be possible to satisfy the homogeneous boundary conditions (14). If we denote by $T_{rl}^{\ j}$, $T_{\theta l}^{\ j}$, $\Phi_l^{\ j}$ $(j = 1, 3)$ the independent surface values of the functions in (14) then if and only if

$$S = \det\,(T_{rl}^{\ j},\ T_{\theta l}^{\ j},\ \Phi_l^{\ j}) = 0 \tag{15}$$

is it possible to construct a non-trivial solution to the problem. The values of s for which (15) is valid constitute a discrete set of eigenvalues of equation (8) subject to boundary conditions (15) and the eigenfunctions associated with each eigenvalue may be determined to within an arbitrary multiplicative constant. The (perhaps infinite) set of eigenvalues for the homogeneous problem is just the set of $s_j^{\ l}$ which appears in the approximate solution (13) except that they are not now selected arbitrarily on the basis of the shape of the spectra $h_l(s)$ of the inhomogeneous problem but rather are determined exactly as the zeros of the secular function (15) of the associated homogeneous problem. If we can determine the set $s_j^{\ l} \equiv \bar{s}_j^{\ l}$ then equation (13) which is the approximate solution of the inhomogeneous problem has the exact

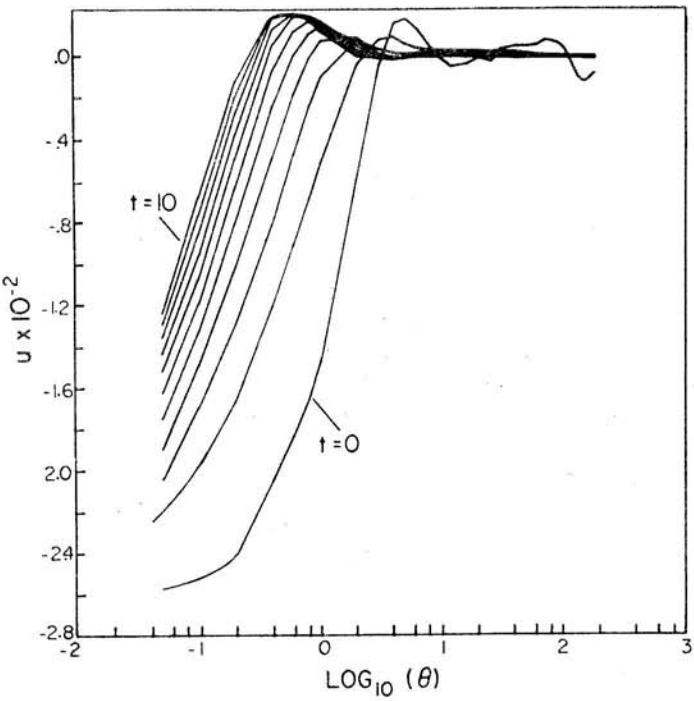

FIG. 9. Same as Fig. 8 but for viscosity model 2.

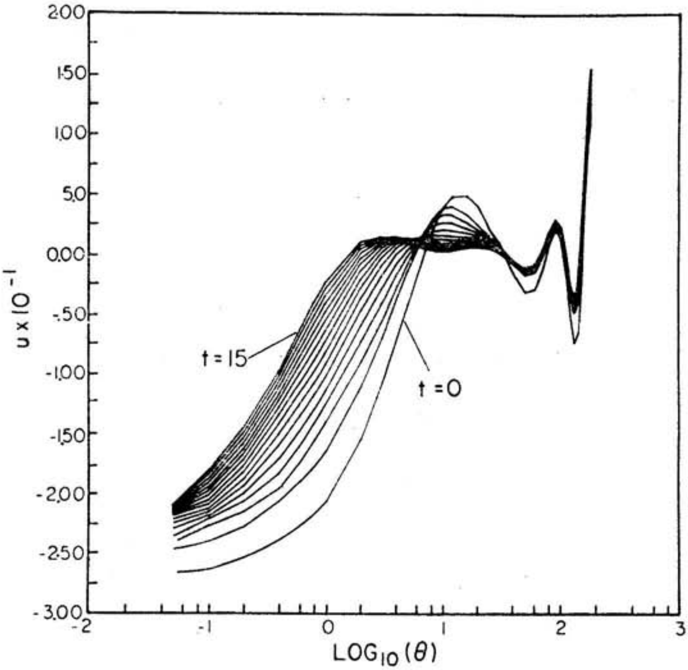

FIG. 10. Same as Fig. 8 but for viscosity model 3.

equivalent form (similarly for the viscous parts of the remaining Love numbers k_l^V, and l_l^V)

$$h_l^V(s) \equiv \sum_j \frac{\bar{r}_j^{\,l}}{s+\bar{s}_j^{\,l}} \tag{16}$$

where the $\bar{s}_j^{\,l}$ are the eigenvalues of the associated homogeneous problem (the poles of the s-spectrum for the inhomogeneous problem) and $\bar{r}_j^{\,l}$ is the residue at the pole $\bar{s}_j^{\,l}$. Equation (16) is in fact a normal mode expansion for the viscous part of the surface load Love number h_l. Gilbert (1970) has shown that the surface load Love numbers in static elasticity have a normal mode expansion in terms of the zero frequency limit of normal modes of elastic-gravitational oscillation of the earth. Here we see that the viscous parts of the surface load Love numbers first introduced in Paper 1 for the visco-elastic problem have an equivalent expansion in terms of the ' normal modes of viscous gravitational relaxation '.

In Fig. 11 we show what shall be referred to as a ' relaxation diagram ' for model 1 which has a uniform viscosity of 10^{22} P from the Earth's surface to the core–mantle boundary and an inviscid core. It is a plot of the eigenvalues of the homogeneous problem $\bar{s}_j^{\,l}$ as a function of Legendre degree l. Each eigenvalue $\bar{s}_j^{\,l}$ has an associated relaxation time $\bar{\tau}_j^{\,l} = 1/\bar{s}_j^{\,l}$ and for each l we have identified a series of modes of relaxation j. This series of modes of relaxation j contains members of three distinct families which we denote by C, M and T. These refer explicitly to the family of core modes, the family of mantle modes and the family of elastico-viscous transition modes. The first two of these families are associated with the two principal density discontinuities in the planetary model; namely that at the core–mantle interface and that at the Earth's surface. These two families each consist of a fundamental mode plus an infinite sequence of ' overtones ' associated with the stratification. The

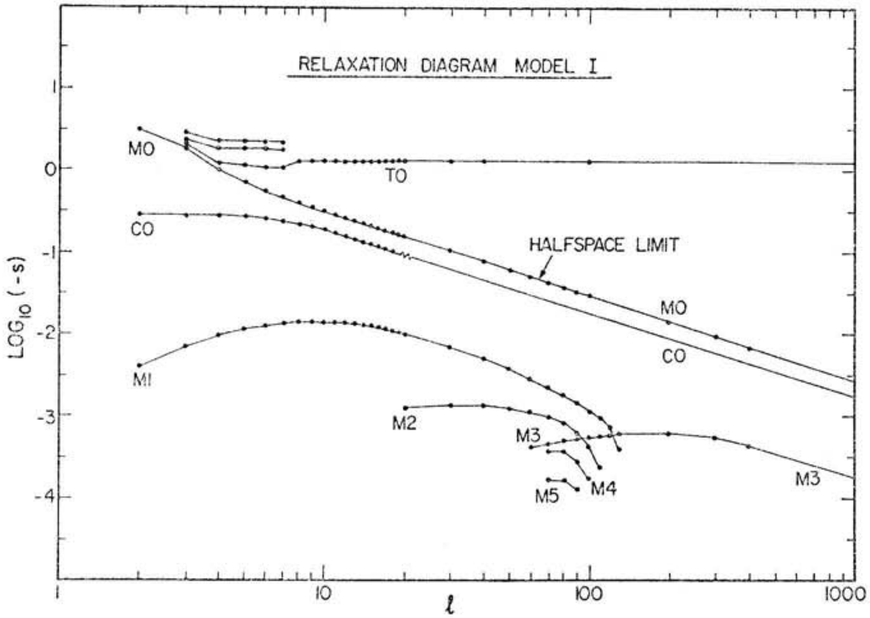

FIG. 11. Relaxation diagram for viscosity model 1. Note the linearity of the relation for the mode M0 for l sufficiently large. This is the half-space limit marked on the figure.

fundamental modes are denoted by C0 and M0 respectively and the overtone sequence by M1, M2 ...; C1, C2 .... In the seven decade range of relaxation times which we have investigated we have been able to locate only a few of the mantle overtones and none of the overtones which should be associated with the core sequence. The family of T modes is associated with the transition from elastic to Newtonian viscous behaviour. As might be expected the members of this family have associated relaxation times which are short compared to those associated with members of the other two groups. Parsons (1971) has previously computed approximations to the fundamental modes C0 and M0 based upon an extension of Darwin's (1879) analysis of the uniform viscous sphere under the assumption that both the mantle and core have uniform density. Here we have accurately determined these modes for a full density stratified real earth model and have in addition been able to calculate their most important overtones. The T modes have not been previously reported.

There are several additional features of the relaxation diagram that warrant discussion. Firstly the break in the mode C0 at $l = 19$ is an artifact of the numerical analysis. Beyond $l = 18$ the starting depth for the numerical integration of (8) is above the core mantle boundary. Since it is this boundary which gives rise to the mode it is not surprizing that it is very poorly determined numerically when the integration does not pass through the boundary itself. The second feature of note is the fact that, for sufficiently large l, the graph of $\log_{10}(-s)$ *vs* $\log l$ tends towards a straight line. This is precisely the result which one should obtain in the uniform half-space limit for the dependence of relaxation time upon wavelength (McConnell 1965). In the limit of small θ with l large we can find the half-space wavenumber equivalent to l using the asymptotic relation

$$P_l(\cos\theta) \to J_0[(l+\tfrac{1}{2})\theta]. \tag{17}$$

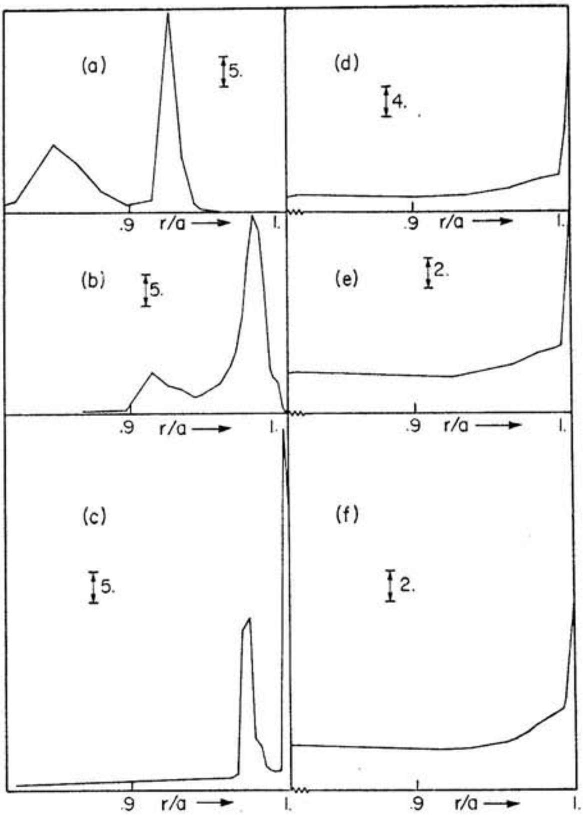

FIG. 12. Plots of the radial dependence of the normalized square of the strain deviator for modes located along the branch M0 in Fig. 11. The curves (a)–(f) correspond respectively to $l = 2, 7$ inclusive. Note that as l increases the shear energy density is progressively more highly concentrated in the near surface region.

The co-ordinate distance θ in a spherical co-ordinate system becomes equal to r/a in the plane cylindrical co-ordinate system where a is the radius of the Earth and where the tangent plane for the cylindrical system touches the Earth's surface at $\theta = 0$. In the cylindrical system the horizontal wavenumber $k_H = (l+\frac{1}{2})/a$. McConnell (1965) and earlier authors have shown that for the half space of constant density ρ_0 and gravitational acceleration g the relaxation time $\bar{\tau}_j{}^l = 1/\bar{s}_j{}^l$ is such that

$$\frac{1}{\bar{s}_j{}^l} = \frac{2\nu k_H}{\rho_0 g} \tag{18}$$

thus

$$\log \bar{s}_j{}^l = -\log k_H$$

and the plot of $\log \bar{s}_j{}^l$ *vs* $\log k_H$ (or $\log(l)$ for sufficiently large l) should be a straight line with a slope of -1. This is obviously true for the M0 mode in Fig. 11.

In Figs 12, 13 and 14 we plot a function which is diagnostic of the mode type for several values of l along the M0, C0 and M1 branches. This function is essentially the radially dependent and appropriately normalized square of the strain deviator Δ_{ij} for the particular mode in question. The function Δ_{ij} is defined as

$$\Delta_{ij} = \tfrac{1}{2}(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i) - \tfrac{1}{3}\partial_k \tilde{u}_k \delta_{ij}. \tag{19}$$

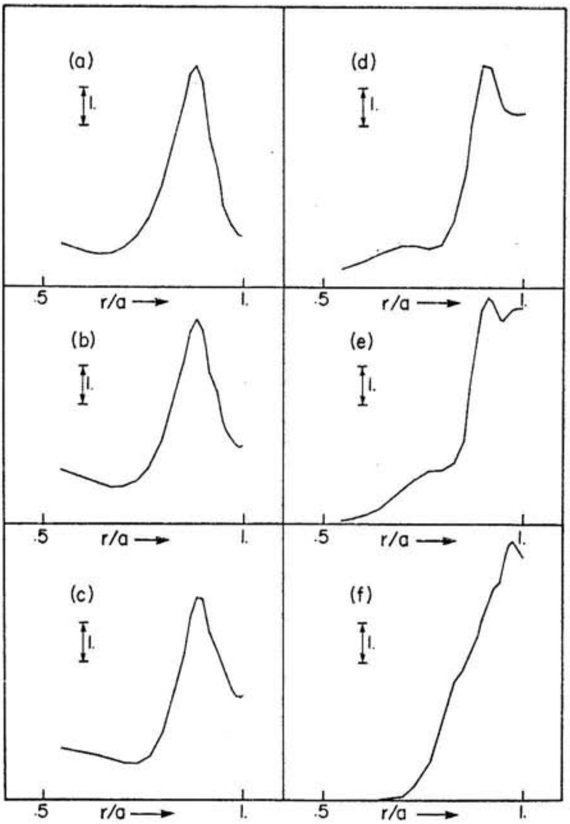

FIG. 13. Same as in Fig. 12 except along the branch M1.

Its square $\Delta_{ij}.\Delta_{ij}$ has a radially dependent part which may be calculated from the normal mode eigenfunctions (see Section 5). It is intimately related to the shear energy distribution in the viscous gravitational relaxation modes just as it is intimately related to the shear energy in the elastic gravitational normal modes of free oscillation. The function $\Delta_{ij}\,\Delta_{ij}$ also plays an important role in determining the differential kernels in the inverse problem for viscosity as we shall see. Inspection of this function in Figs 12–14 immediately explains our particular choice for the mode labelling which we have employed. The fundamental core mode has its shear energy concentrated at the core–mantle interface while the fundamental mantle mode has its shear energy concentrated near the Earth's surface. The overtone mode M1 has a radial structure similar to M0 but its shear energy is not so highly confined in the near surface region. The T modes have radial structure functions which are similar to the M modes.

Because of the complex mode structure apparent in Fig. 11 even for a mantle with uniform viscosity we may well be concerned as to the validity of previous half-space models of the relaxation process. In such models the fundamental core mode is certainly absent, and, since such models are usually assumed to have uniform density, the sequence of mantle overtones is absent also. The extent to which these modes are important in the inhomogeneous problem is of course determined by the residues $\bar{r}_j^l$ which are associated with them, or, via the analysis in Gilbert (1970) of the relative projection of the forcing upon the respective members of the sequence of normal mode eigenfunctions. However, we have already solved the inhomogeneous problem (albeit approximately) and so we have a good qualitative feel for the relative import-

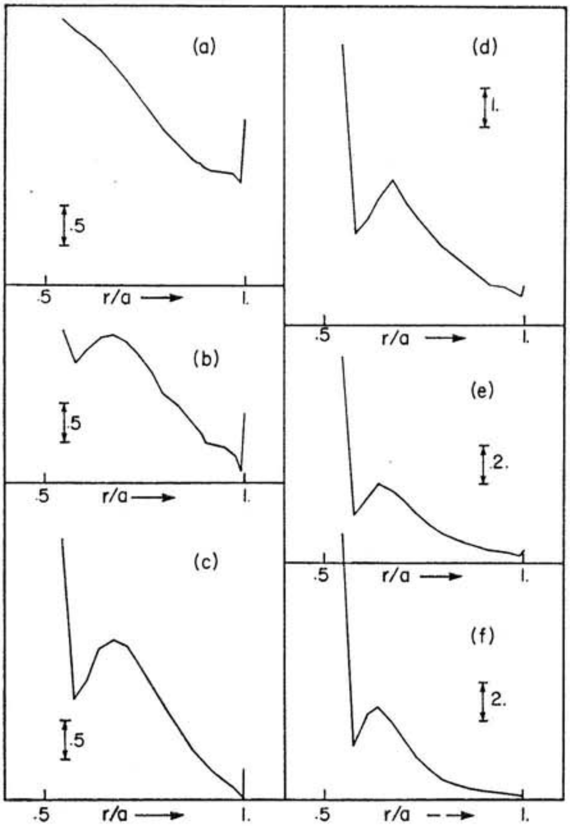

FIG. 14. Same as Fig. 12 except along the branch C0 in Fig. 11.

ance of the various modes as a function of l. Inspection of $h_l^V(t)$ for model 1 in Fig. 5 shows that there are at least two important modes in this highly non-exponential time history for sufficiently small l. Comparing Fig. 5 with Fig. 11 immediately confirms that both C0 and M0 are excited for this viscosity model. Furthermore, it is not until $l \lesssim 10$ that the decay curves become strictly exponential. Thus using (17) we can see that if we were to employ a half-space model with this viscosity to fit a relaxation curve then the half-space model would be highly inaccurate for wave-numbers less than about $2 \times 10^{-8}\,\mathrm{cm}^{-1}$. We will see shortly that this is the source of a fundamental error in McConnell's (1965) analysis which led him to believe that the Earth's mantle must have a viscosity which increases with depth. The presence of the core and overtone modes as well as the fundamental mantle mode in the spherical self-gravitating models is, on the other hand, the feature which allows such models to fit the observed relaxation curves even with uniform mantle viscosity.

In Figs 15 and 16 we show examples of two further relaxation diagrams to illustrate respectively, the effect of a high viscosity in the lower mantle and of a lithospheric lid. Fig. 15 is the relaxation diagram for model 3 and Fig. 16 is the relaxation diagram for model 4. Fig. 15 demonstrates that the effect of the high viscosity lower mantle is to move the relaxation times for all modes with l sufficiently small to much longer times. For l sufficiently large the relaxation times are unaffected. This fact is strikingly evident also in the approximate Love number time histories calculated previously for the forced problem. Comparison of Figs 5 and 7 illustrates that the Love numbers for small l have very much longer decay times for the high lower mantle viscosity

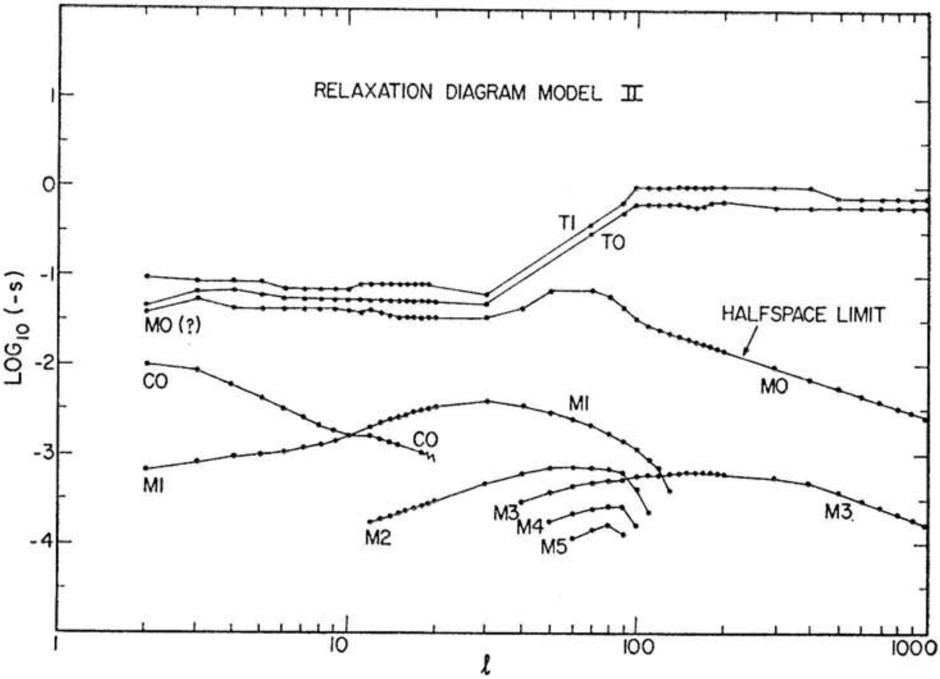

FIG. 15. Relaxation diagram for viscosity model 3. Note the increase in decay times for modes with small l (large wavelength) from those in Fig. 11. The relaxation times for the large l modes remain unchanged.

model than they do in the uniform case. Inspection of Fig. 15 seems also to confirm the idea expressed previously in Paper 1 that the T modes are more important in this model than they were found to be in the uniform case. In Fig. 15 the transition modes are seen to be in much closer proximity to M0 than before, and indeed, M0 for small l is rather like a transition mode.

Introduction of a lithosphere into the uniform model with a thickness of 120 km and a viscosity of 10^{25} P (cgs) effects a pronounced change in the relaxation diagram for large values of l as can be seen by inspection of Fig. 16. For M0 the relaxation time increases to $l = 30$ then decreases sharply for larger l values. This effect is also evident in McConnell's (1965) half-space calculations. If $\nu = \infty$ in the lithosphere which is the case McConnell treats, then the decrease in decay time for the M0 mode would continue for all $l \lesssim 30$. In the present example with $\nu = 10^{25}$ P in the lithosphere, the curve eventually turns back to the appropriate half-space asymptote on which $\log s_j^l = -\log k_H$ as before although the level of the curve is shifted from that for the uniform 10^{22} P mantle because of the higher near surface viscosity. For small values of $l \gtrsim 6$ both C0 and M0 are exactly as in the uniform case but the first mantle overtone M1 is now found with much shortened decay times than previously. In the range $5 \leqslant l \leqslant 7$ a striking example of mode conversion is apparent. The mode C0 jumps to what was the M1 branch and M1 to what was the C0 branch. This may be confirmed by inspection of the square of the radially dependent strain deviator.

In Fig. 17 we have plotted wavenumber $k_H = (l+\frac{1}{2})/a$ (in the asymptotic limit (17)) as a function of relaxation time for the three principal modes of model 4 (i.e. C0, M0, M1). The open circles are employed by McConnell in his half-space analysis, the interpretation of which led him to conclude that the mantle should have a viscosity which increases (perhaps strongly) as a function of depth. Because of the half-space (constant density) approximation, the only relaxation curve he had to work with was

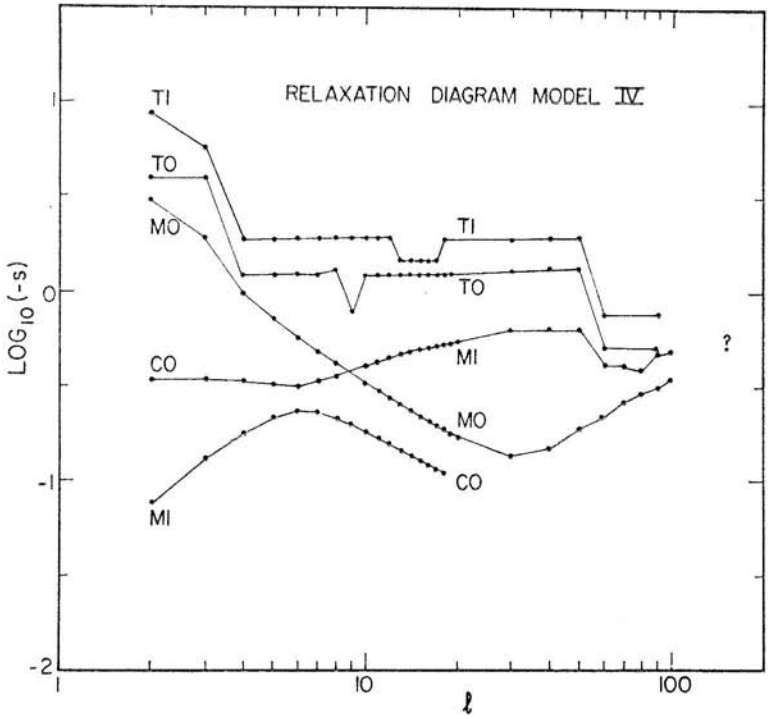

FIG. 16. Partial relaxation diagram for viscosity model 4 which is the same as model 1 except that it has a 120 km thick lithosphere with a viscosity $\nu = 10^{22}$ Poise (cgs). Note the sharp decrease in relaxation time for modes along the M0 branch for $l \geqslant 20$.

our M0. For small wavenumbers, we see from Fig. 17, that M0 has too short a relaxation time to satisfy the data. We can increase the relaxation times for M0 by increasing the deep mantle viscosity and this was the approach McConnell was obliged to take. However, with a spherical self-gravitating model we also have C0 and M1 available to us and Fig. 17 shows that C0 is strongly excited in the uniform mantle case. I believe that this is a fundamental difficulty with McConnell's interpretation. It explains why the spherical self-gravitating models are able to fit the data with a uniform mantle viscosity (Peltier & Andrews 1976).

Further inspection of Fig. 17 demonstrates that the turnover at large wavenumbers which is characteristic of the data is not well fit by the model lithosphere which we have adopted although the character of the M0 curve is certainly similar to that which is observed. We expect that a perfectly elastic lithosphere which is on the order of 120 km thick, similar to that employed by McConnell, would allow us to fit the long wavenumber tail on the relaxation curve with the spherical model just as McConnell was able to fit this feature with his half space model.

We have shown in the above, that there exists a unique relaxation diagram which is characteristic of every linear visco-elastic (Maxwell) model of the interior of the Earth. This diagram plays precisely the same role in the problem of viscous gravitational relaxation as does the dispersion relation in the problem of the elastic gravitational free oscillations. In addition we have seen in equation (16) that the viscous part of the solution to the inhomogeneous problem has the form of an expansion in terms of the normal modes of relaxation. As stated previously this is the property of the physical system which enables us to construct a simple first-order perturbation theory for the determination of the viscosity in the deep interior of the

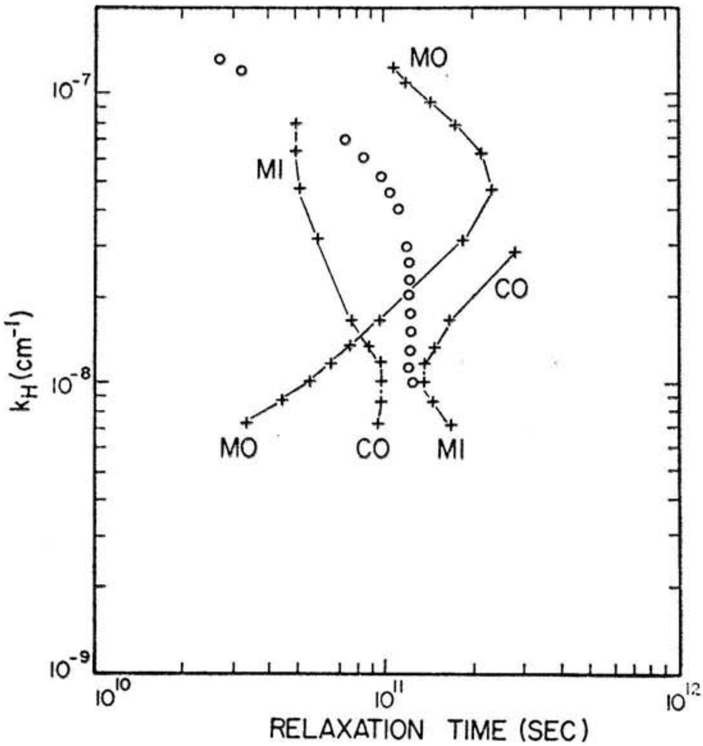

FIG. 17. Equivalent wavenumber *vs* relaxation time plot for the dominant modes of viscosity model 4 (crosses). Superimposed upon this theoretical data are the observed relaxation times as a function of wavenumber deduced by McConnell from the Fennoscandia data set.

planet. This theory is constructed and verified in the succeeding sections. We will employ it in future to refine the viscosity models previously obtained by model fitting.

3. Statement of the forward problem

Suppose we know that at some time in the past (say $t = 0$) the surface distribution of ice and water was in gravitational equilibrium with the underlying planet. Subsequent to this time we suppose further that the total surface mass load began to undergo a redistribution in response to a climatic change. The assumption of an initial gravitational equilibrium is precisely equivalent to the assumption of an initial ' isostatic ' equilibrium. If we knew with absolute accuracy the temporal and spatial details of this load redistribution process and if we knew with the same confidence that the planet could be described as a radially stratified Maxwell solid with known density, Lamé constants, and Newtonian viscosity, then we could calculate exactly the response of the planet to the redistribution of surface mass. This is subject of course to the additional qualification that the strains produced by unloading are small enough that the linearized field equations remain valid. For example; if we wish to calculate the time dependence of the absolute radius at a particular co-latitude and east longitude point on the surface with co-ordinates (θ, ϕ) then we simply convolve the Green function in equation (12) with the known load as

$$\Delta R(\theta, \phi, t) = \int dt' \int d\Omega' \, G_R(\gamma, t-t') \, M(\theta, \phi, t') \, H(t') \tag{20}$$

where ΔR is the change in absolute radius, $d\Omega'$ is the element of area on the assumed spherical surface and the angle γ is the angular separation between the 'source point' (θ', ϕ') and the 'field point' (θ, ϕ). Equation (20) is the equation we must be able to solve in order to do the forward problem. Of course (20) describes only one signature of the response. It could also be described in terms of its gravity anomaly, or surface tilt or by the time dependent separation between adjacent quasi-spherical equi-potential surfaces (Farrell & Clark 1976, companion paper). Any of these signatures may be determined by solving an equation like (20). We may choose to do the convolution integral on the right-hand side spectrally by decomposing the load $M(\theta, \phi, t)$ into its time dependent surface spherical harmonic coefficients or we may choose to do it in grid point space. The latter approach was taken by Peltier & Andrews (1976).

By assuming that the time dependence of the load is step discontinuous the convolution over time in (20) may be done analytically. This generates a new set of time dependent Green functions which I have called Heaviside Green functions. These new Green functions have been described in Peltier & Andrews (1976).

4. Statement of the inverse problem

In general the inverse problem for any signature of the isostatic adjustment process is an awkward one because of the structure of the forward problem (20). In general, we know neither the visco-elastic structure of the interior (which is contained in the Green function), nor do we know the precise space time dependence of the surface mass load. Therefore the problem is a highly non-linear one. If we suppose that the radial variations of the Earth's density and of its Lamé constants are determined exactly by the free oscillations data then the model has only two unknown functionals. These two functionals are the surface mass load (a function of position on the Earth's surface and of time) and the radially dependent Newtonian viscosity. The former I shall refer to as the mass function 'M' and the latter as the viscosity functional 'V'. With both M and V unknown we are forced to adopt an iterative approach to the resolution of the inverse problem. The space time dependent surface loads at the end of the Pleistocene period have been described to zeroth order by Patterson (1972) for the Laurentide region and by previous authors for the Fennoscandia area. There are three main pieces of information required to produce such a description of M. These are:

(a) The so-called 'eustatic curves' give the total mass influx of water into the ocean basins as a function of time. (See the companion paper by Farrell and Clark for a clear redefinition of this notion.)

(b) End moraine data provide isochrons on the time dependent position of the edges of the major ice sheets.

(c) Ice mechanics (Patterson 1972) gives a theoretical prediction of the way in which a given ice volume is distributed over a given surface area.

Errors in the application of all of (a)–(c) make the description of M inexact, however we expect that the actual M does not differ too much from the M calculated in this fashion. We make use of this prior knowledge of M to provide an initial linearization of the inverse problem. That is, we make the initial assumption that M is precisely known. Now M consists of two parts, these being M_I which is the 'negative' load applied by deglaciation and M_0 the positive load applied by the simultaneous addition of melt water to the ocean basins. Since mass must be conserved in this loading process if the hydrological cycle is closed we must have

$$M_{\mathrm{I}}+M_0 \equiv 0. \tag{21}$$

Equation (21) is the basis of (a) above. However, given M_{I} we can in fact calculate M_0 exactly not only with respect to its magnitude as has always been done in the past but also with respect to the way in which M_0 is distributed over the known surface of the ocean basins. To do this we simply demand that the oceans' surface remains an equipotential surface. Then M_0 is determined by inverting an integral equation (Farrell & Clarke 1976). Thus the only unknown part of M is the ice functional M_{I} and this we can approximately determine using (a)–(c).

Given M approximately determined in the above fashion we may write the solution (20) to the forward problem in the Laplace transform domain by applying the convolution theorem as

$$\Delta R(\theta, \phi, s) = \int d\Omega' \, G_R(\gamma, s) \, M(\theta', \phi', s). \tag{22}$$

The unknown viscosity functional V is now imbedded in G_R. This solution of the forward problem is now linear in V with M given. In order to solve the inverse problem for the determination of V given $\Delta R(\theta, \phi, s)$ we must determine the sensitivity of the change in radius ΔR to small spatially dependent changes in the Newtonian viscosity ν, say $\delta\nu$, where $\nu = \nu_0(r)+\delta\nu$. We apply the formalism of the calculus of variations to calculate this sensitivity. The first variation of equation (22) is just:

$$\delta\Delta R(\theta, \phi, s) = \int d\Omega' \, \delta G_R(\gamma, s) \, M(\theta', \phi', s). \tag{23}$$

where $\delta M \equiv 0$ since M is assumed known. Now using the definition (12) we may write δG_R as

$$\delta G_R(\gamma, s) = \frac{a}{m_e} \sum_{l=0}^{\infty} \delta h_l^V(s) \, P_l(\cos\gamma) \tag{24}$$

since $\delta h_l^E \equiv 0$ because we assume that the Lamé constants and density are known exactly from the free oscillations data. The main problem we have to face here is the problem of showing that the functional $\delta h_l^V(s)$ evaluated at fixed s may be determined exactly once the small variation $\delta\nu$ in an initial viscosity profile $\nu_0(r)$ is specified. To proceed further we invoke the normal mode expansion of h_l^V which in the Laplace transform domain has the form (16), i.e.

$$h_l^V(s) = \sum_j \frac{\bar{r}_j^{\,l}}{s+\bar{s}_j^{\,l}}. \tag{25}$$

The first variation of $h_l^V(s)$ at fixed s may be found from (25) as

$$\delta h_l^V(s) = \sum_j \frac{\partial h_l(s)}{\partial \bar{s}_j^{\,l}} \delta\bar{s}_j^{\,l} + \sum_j \frac{\partial h_l(s)}{\partial \bar{r}_j^{\,l}} \delta\bar{r}_j^{\,l}. \tag{26}$$

Thus both the positions of the poles $\bar{s}_j^{\,l}$ and their associated residues $\bar{r}_j^{\,l}$ are affected by the variation of the viscosity $\delta\nu$. The shift in the free decay pole of the relaxation spectrum is $\delta\bar{s}_j^{\,l}$ and the associated shift in its residue is $\delta\bar{r}_j^{\,l}$. Only if we can show that *both* $\delta\bar{s}_j^{\,l}$ and $\delta\bar{r}_j^{\,l}$ are expressible in terms of some weighted average of the viscosity variation $\delta\nu$ will it be possible to close the inverse theory for mantle viscosity with a first-order perturbation analysis. The partial derivative terms in (26) may, of course, be determined by direct differentiation of (25) as

$$\frac{\partial h_l^V(s)}{\partial \bar{s}_j^{\,l}} = -\frac{\bar{r}_j^{\,l}}{(s+\bar{s}_j^{\,l})^2} \tag{27}$$

$$\frac{\partial h_l^V(s)}{\partial \bar{r}_j^{\,l}} = \frac{1}{(s+\bar{s}_j^{\,l})}. \tag{28}$$

From equation (22) we can pose a second inverse problem, besides that for mantle viscosity which we have considered above. If we were in a position to state that the viscosity of the mantle were known with greater accuracy than the distribution of surface mass functional M, we could then formulate an inverse problem for M rather than for the viscosity functional V. Again taking first order variations in (22) we obtain

$$\delta \Delta R(\theta, \phi, s) = \int d\Omega' \, G_R(\gamma, s) \, \delta M(\theta', \phi', s). \tag{29}$$

Therefore the differential kernels for the refinement of the mass functional are just the surface load Green functions themselves. Equations like (29) determine the sensitivity of the response to small variations in M.

If we can determine $\delta h_l^V(s)$ then we may proceed iteratively to refine our knowledge of both V and M. We first fix M and refine V. We then fix V and refine M, and hope that the process of sequential refinement converges. This method of attack upon an essentially non-linear problem has an exact analogy with the recent work of Gilbert & Dziewonski (1975) on the inversion of the free oscillation data set. In their problem there are also two distinct unknown components of the model system. The first of these being the elastic structure of the interior and the second the moment tensor of the earthquake which gave rise to the observed free oscillations. In the present problem the viscosity functional V is analogous to their elasticity structure and the mass functional M to their moment tensor.

In the next section we give the relationship between the variation in viscosity $\delta\nu$ and the shift in a free decay pole of the relaxation spectrum as required in equation (26).

5. Differential kernel for the shift in a free decay pole

From the field equations (6) and (7) and the Laplace transformed constitutive relation (2) we can derive a variational principle for the homogeneous problem. This is exactly analogous to Rayleigh's variational principle of elasticity. It states (see Appendix A) that to first order the following integral relation is valid.

$$\int_V dv[2\Delta_{ij} \Delta_{ij} \delta\mu(s)] = 0 \tag{30}$$

where Δ_{ij} is the Laplace transformed strain deviator defined in equation (19) and where $\mu(s)$ is the compliance defined in equation (4). The volume V is the Earth's volume. From equation (30) we can immediately deduce the expression for the differential kernel relating the shift in a free decay pole of the relaxation spectrum to a (small) change in the viscosity model $\delta\nu$. We note from (4) that $\mu(s)$ is a function not only of the position of the pole s but also of the viscosity ν. It is a function only of these parameters since the Lamé constant μ is assumed fixed. Thus for the variation of $\mu(s)$, $\delta\mu(s)$, we may write

$$\delta\mu(s) = \left(\frac{\partial\mu}{\partial\nu}\right)_s \delta\nu + \left(\frac{\partial\mu}{\partial s}\right)_\nu \delta s \tag{31}$$

but from equation (4)

$$\left(\frac{\partial\mu(s)}{\partial\nu}\right)_s = \frac{\mu^2 s/\nu^2}{(s+(\mu/\nu))^2} \tag{32}$$

$$\left(\frac{\partial\mu(s)}{\partial s}\right)_\nu = \frac{\mu(\mu/\nu)}{(s+(\mu/\nu))^2}. \tag{33}$$

Both of the derivatives (32) and (33) are positive definite, a fact which is important to the structure of the differential kernel which we shall obtain. Substituting (31) into (30) gives

$$\int_V dv \left[\Delta_{ij}\,\Delta_{ij}\left(\left(\frac{\partial\mu(s)}{\partial v}\right)_s \delta v + \left(\frac{\partial\mu(s)}{\partial s}\right)_v \delta s\right)\right] = 0. \tag{34}$$

Since neither s nor δs are functions of position we may extract them both from under the integral sign and write

$$\frac{\delta s}{s} = \frac{-\int_V dv\left[\Delta_{ij}\,\Delta_{ij}\left(\frac{\partial\mu(s)}{\partial v}\right)_s \frac{\delta v}{s}\right]}{\int_V dv\left[\Delta_{ij}\,\Delta_{ij}\left(\frac{\partial\mu(s)}{\partial s}\right)_v\right]}. \tag{35}$$

This is a general expression for the shift in a free decay pole which we have been seeking. Given δv we can compute δs. Nowhere in the derivation of (35) have we made any explicit restriction to the consideration of radial variations of viscosity only. Equation (35) applies equally well to variations of viscosity which are arbitrary functions of position within the spherical body so long as these variations are small. The viscosity of the background state for which Δ_{ij} is computed must of course be radially stratified. The only previous attempt to derive a variational relation like (35) of which I am aware is the work of Parsons (1972) which dealt with the non-gravitating viscous half-space problem. The kernel which we have derived has exactly the same principal part as the one obtained by Parsons (namely $\Delta_{ij}\,\Delta_{ij}$), but it determines the shift in an arbitrary pole of the relaxation spectrum rather than the change in the unique relaxation time for some particular wavenumber of the deformation. In the spherical problem, as we have seen, any fixed harmonic has associated with it a discrete set of relaxation times and equation (35) can be employed to determine the shift of any one of these which is produced by a particular viscosity variation.

If we restrict our attention to variations of viscosity $\delta v(r)$ which are functions of radius only then the problem of computing the kernels in (35) is reduced considerably. We employ Backus' (1967) result to the effect that for any vector field $\tilde{\mathbf{u}}$ defined in $0 \leqslant r \leqslant a$ there are unique scalar fields $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ such that V and W average to zero on every spherical surface concentric with the origin. These fields are defined as

$$\tilde{\mathbf{u}} = \tilde{U}\hat{\mathbf{r}} + \nabla_1 \tilde{V} - \hat{\mathbf{r}} \wedge \nabla_1 \tilde{W} \tag{36}$$

with $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ unit vectors in the directions of increasing radius $\hat{r}$, co-latitude θ and longitude ϕ and

$$\nabla_1 = \hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \operatorname{cosec}\theta\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\phi}. \tag{37}$$

Solutions of the spheroidal equations are such that $\tilde{W} \equiv 0$ and the functions $\tilde{U}$ and $\tilde{V}$ are just the radial and tangential displacements which we calculate as solutions to the field equations. These functions, together with $\tilde{\phi}$ are all products of the form $\tilde{U}_l(r)\,Y_{lm}(\theta,\phi)$, $\tilde{V}_l(r)\,Y_{lm}(\theta,\phi)$, $\tilde{\phi}_l\,Y_{lm}(\theta,\phi)$ where $Y_{lm}(\theta,\phi)$ is a normalized surface spherical harmonic. Backus & Gilbert (1968) describe in detail the reduction of the kernels for the elastic gravitational free oscillations problem which are similar to that in equation (35). Their arguments follow through exactly in the present case and we find that with $\delta\nu = \delta\nu(r)$ equation (35) reduces to the following form

$$\frac{\delta s}{s} = \frac{-\int_b^a dr\, r^2 K_l\left(\frac{\partial\mu(s)}{\partial s}\right)_s \frac{\delta\nu(r)}{s}}{\int_b^a dr\, r^2 K_l\left(\frac{\partial\mu(s)}{\partial s}\right)_\nu}. \tag{38}$$

The function K_l is given by

$$K_l = \tfrac{1}{3}(2\partial_r U_l - F)^2 + (1/r^2)\, l(l+1)\,(r\partial_r V_l - V_l + U_l)^2 + (1/r^2)\,(l-1)\, l(l+1)\,(l+2)\, V_l^2 \tag{39}$$

where

$$F = (1/r)\, 2U_l - l(l+1)\, V_l. \tag{40}$$

In (38) the integration extends from the core–mantle boundary $r = b$ to the Earth's surface $r = a$. The integral over the core vanishes because $\mu(s) \equiv 0$ there. It should be recalled that the functions U_l and V_l appearing in (39) and (40) are eigenfunctions of the homogeneous problem and that (38) is thus valid only in the vicinity of an eigenvalue $s = \bar{s}_j^{\,l}$. Explicit substitution of the partial derivatives (32) and (33) in equation (38) gives

$$\frac{\delta\bar{s}_j^{\,l}}{\bar{s}_j^{\,l}} = \frac{-\int_b^a dr\, r^2\left[K_l \frac{\mu^2/\nu}{(\bar{s}_j^{\,l} + (\mu/\nu))^2}\right]\frac{\delta\nu}{\nu}}{\int_b^a dr\, r^2\left[K_l \frac{\mu^2/\nu}{(\bar{s}_j^{\,l} + (\mu/\nu))^2}\right]}. \tag{41}$$

If we define x and y such that

$$\left.\begin{aligned} \bar{s}_j^{\,l} &= 10^x \\ \nu &= 10^y \end{aligned}\right\} \tag{42}$$

then

$$\left.\begin{aligned} \delta x &= \delta\bar{s}_j^{\,l}/\bar{s}_j^{\,l} \\ \delta y &= \delta\nu/\nu \end{aligned}\right\} \tag{43}$$

and we see clearly revealed the intrinsic logarithmic scaling of the relaxation problem. Equation (41) thus determines the relationship between the shift in the radially dependent *exponent* of the viscosity model and the associated shift in the *exponent* of a free decay pole in the relaxation spectrum. Because of the minus sign in (41) we see that as the viscosity is perturbed to larger values then the decay time $\bar{\tau}_j^{\,l} = 1/\bar{s}_j^{\,l}$ increases. The exponent relation is such that if $\delta y = +1$ then $\delta x = -1$, a very simple result.

We have previously plotted in Figs 12–14 a series of differential kernels for the shift in a free decay pole for all of M0, C0 and M1 and for a variety of l values. We have seen previously that these were highly diagnostic of the particular family of poles to which a particular pole belonged. The reason for this intimate relation is of course due to the connection between $\Delta_{ij}\,\Delta_{ij}$ and the shear energy in the mode.

The relation (41) has been tested by assuming a variety of radially distributed viscosity variations described through $y(r)$ and computing the pole shift in the first-order variational formula (41). The predicted pole shifts are then compared with

Table 1

y_0	$\dfrac{\delta x}{\delta \bar{s}^2_{M1}}$ (first order)	$\dfrac{\delta x}{\delta \bar{s}^2_{M1}}$ (actual)	% error
0·005	−2·364–3	−2·343–3	<1
0·01	−4·729–3	−4·690–3	<1
0·05	−2·364–2	−2·500–2	5·7
0·1	−4·729–2	−5·000–2	5·4
0·5	−2·364–1	−2·258–1	4·8
1·0	−4·729–1	−3·660–1	29·0

the exact pole positions calculated for the new viscosity model. We assumed a simple functional representation

$$\delta y(r) = y_0 \cos\left[\Pi(r-b)/(a-b)\right] \tag{44}$$

with a range of perturbation amplitudes y_0. The results of this test are listed in Table 1 for the M1 mode at $l = 2$. It will be seen by inspection of these results that the problem is highly linear. Even with viscosity variations through the mantle which are as large as 10^2 (with $y_0 = 1$) the prediction of the new pole location based on the variational principle is surprisingly accurate although this accuracy certainly decreases as y_0 increases, the largest error being for $y_0 = 1$ and having a magnitude equal to 30 per cent of the real shift in y. This is an extremely encouraging result which leads us to expect that the linear inverse theory will be very rapidly convergent. These linearity tests will be discussed in greater detail in the future.

6. An integral constraint relating the shift in a free decay pole to the shift in its associated residue

In Section 4, it was shown that we had to determine the shift in the residue $\bar{r}_j^{\,l}$ associated with the shift in the jth pole $\bar{s}_j^{\,l}$ in terms of $\delta\bar{s}_j^{\,l}$ in order to close the theory in a linear perturbation sense. Consider any one of the Love number s-spectra illustrated in Section 2 for the solution of the inhomogeneous problem. Their viscous parts all have an exact normal mode expansion in the form (16) which we reproduce here for convenience as

$$h_l^{V}(s) = \sum_j \frac{\bar{r}_j^{\,l}}{s+\bar{s}_j^{\,l}}. \tag{45}$$

Consider the limit of (45) as $s \to 0$. This is just

$$\lim_{s\to 0} h_l^{V}(s) = \sum_j \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}}. \tag{46}$$

This function has a rather important physical meaning which was alluded to in Paper 1. To appreciate this consider the $h_l^{V}(t)$ corresponding to (45). This has the simple form

$$h_l^{V}(t) = \sum_j \bar{r}_j^{\,l} \exp\left(-\bar{s}_j^{\,l} t\right). \tag{47}$$

Suppose that the system (planet) were driven by a constant surface mass load applied at $t = 0$ and maintained. The harmonic coefficients of the resulting temporal response are obtained by convolving their impulse response forms (47) with a Heaviside step function. These time dependent coefficients may be called Heaviside Love numbers and they have the form

$$\begin{aligned} h_l^{H}(t) &= h_l(t) * H(t) \\ &= \sum_j \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}}\left(1-\exp\left(-\bar{s}_j^{\,l} t\right)\right) + h_l^{E}. \end{aligned} \tag{48}$$

These have been employed in Peltier & Andrews (1976) to synthesize Heaviside Green functions while constructing solutions to the forward problem. If we strip off the elastic response as usual by writing

$$h_l^{H,V}(t) = h_l^{H}(t) - h_l^{E} \tag{49}$$

$$h_l^{H,V}(t) = \sum_j \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}} \left(1 - \exp\left(-\bar{s}_j^{\,l} t\right)\right). \tag{50}$$

It will now be seen by inspection of equation (50) that the final amplitude of $h_l^{H,V}(t)$ is just

$$\lim_{t \to \infty} h_l^{H,V}(t) = \sum_j \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}}$$

$$= \lim_{s \to 0} h_l^{V}(s). \tag{51}$$

Therefore the summation in equation (51) determines the maximum amplitude of the harmonic component of order l when the planet relaxes in response to gravitational interaction with a unit point mass applied to its surface and maintained. Of course this is only the viscous part of the total distortion – but the elastic part is simply an additive constant. This final amplitude is just the isostatically adjusted amplitude of the particular harmonic in question. This isostatically adjusted amplitude may therefore be read directly as the small s asymptote of the Love number s-spectrum for that harmonic according to the equivalence expressed by equation (51). Furthermore, since the Maxwell solid is viscously incompressible no variations in density are produced in the viscous relaxation process. Thus the final isostatically adjusted state for each harmonic *is unique and independent of the particular viscosity model of the interior which we have chosen* (M constant). That this is indeed so may be confirmed by direct inspection of the Love number s-spectra shown in Section 2 for three widely different viscosity models. This conclusion may also be reached by argument from first principles but the exercise is straightforward and we will not give it here. The net effect of adjustment for a Maxwell solid is simply a change in the planet's shape.

Although the time taken for a particular model to reach its final state of isostatic equilibrium may vary widely from model to model this final state is inevitably the same state so long as M is fixed. Therefore, there exists a constraint on all viscoelastic (Maxwell) models of the interior such that the parameter

$$\mathscr{L}_l = \sum_j \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}} \tag{52}$$

is a constant for fixed l. This physical constraint is the factor which makes it possible to rigorously invert the inhomogeneous problem. In equation (26) we require a relation between the shift in the jth residue $\bar{r}_j^{\,l}$ for degree l and the shift in the pole $(-)\,\delta\bar{s}_j^{\,l}$ with which it is associated.

To calculate this quantity we suppose that an initial mantle viscosity profile $v(r)$ is subject to a particular variation $\Delta v(r)$ which need not be small. Then $\bar{s}_j^{\,l} \to \bar{s}_j^{\,l} + \Delta\bar{s}_j^{\,l}$ and $\bar{r}_j^{\,l} \to \bar{r}_j^{\,l} + \Delta\bar{r}_j^{\,l}$ but since the isostatic state is unique for a fixed applied load therefore the number $\mathscr{L}_l$ in (52) does not change. Thus

$$\mathscr{L}_l = \sum_j \frac{\bar{r}_j}{\bar{s}_j} = \sum_j \frac{\bar{r}_j^{\,l} + \Delta\bar{r}_j^{\,l}}{\bar{s}_j^{\,l} + \Delta\bar{s}_j^{\,l}}. \tag{53}$$

Now in general equation (53) is an integral constraint on the system and it does not follow immediately in the sum

$$\sum_j \left(\frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}} - \frac{\bar{r}_j^{\,l} + \Delta \bar{r}_j^{\,l}}{\bar{s}_j^{\,l} + \Delta \bar{s}_j^{\,l}} \right) = 0. \tag{54}$$

that the separate terms for each j cancel term by term. Since $\Delta v(r)$ may be arbitrarily large it may force arbitrarily large variations $\Delta \bar{r}_j^{\,l}$ and $\Delta \bar{s}_j^{\,l}$. Thus modes which are only weakly excited in the $v(r)$ state (i.e. have small associated residues) may be very efficiently excited in the perturbed state $v(r) + \Delta v(r)$. However if we restrict our attention to perturbations $\Delta v(r) \equiv \delta v(r)$ which are sufficiently small then we may expand the function $f(\bar{r}_j^{\,l} + \Delta r_j^{\,l}, \bar{s}_j^{\,l} + \Delta \bar{s}_j^{\,l})$ in a two-dimensional Tayler series about its value $f(\bar{r}_j^{\,l}, \bar{s}_j^{\,l})$ in which case we obtain from (54)

$$\sum_j \left(\frac{\bar{r}_j^{\,l}}{(\bar{s}_j^{\,l})^2} \delta \bar{s}_j^{\,l} - \frac{1}{\bar{s}_j^{\,l}} \delta \bar{r}_j^{\,l} \right) = 0$$

in which we may reasonably expect the series to vanish term by term. This leads to the result

$$\delta \bar{r}_j^{\,l} = \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}} \delta \bar{s}_j^{\,l} \tag{55}$$

which is the required relation between the shift in the jth pole and the associated shift in the jth residue. In deriving it we have made use of the fact that for the forced viscous gravitational relaxation of a Maxwell solid the final gravitational equilibrium configuration (isostatic state) is unique.

Substitution of (55) into (26), making use of the partial derivatives (27) and (28) leads to the following expression for the variation of the Love number $h_l(s)$ (similar expressions may also be derived of course for the remaining Love numbers $k_l(s)$ and $l_l(s)$)

$$\delta h_l(s) = \sum_j \left(\frac{-\bar{r}_j^{\,l}}{(s + \bar{s}_j^{\,l})^2} \right) \delta \bar{s}_j^{\,l} + \sum_j \left(\frac{1}{s + \bar{s}_j^{\,l}} \right) \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}} \delta \bar{s}_j^{\,l} = \sum_j \frac{s}{\bar{s}_j^{\,l}} \frac{\bar{r}_j^{\,l}}{(s + \bar{s}_j^{\,l})^2} \delta \bar{s}_j^{\,l}. \tag{56}$$

It is equation (56) which makes rigorous inversion of isostatic adjustment data possible. This equation coupled with equation (51) gives the variation in the surface load Love numbers $h_l(s)$ directly in the form of an integral over the distributed viscosity variation $\delta v(r)$. We are thus in a position to claim to have constructed a complete inverse theory for the forced relaxation problem. There are several elaborations of this theory, however, which are required to meet the demands of the data set which we have available. This data set has been discussed in detail by Peltier & Andrews (1976, companion paper). The main difficulty which we have to face in inverting this observational data arises due to the fact that these data are only sparsely available as individual submerged and/or elevated beach histories in the space domain. This means that we cannot hope to decompose them into their various time dependent spherical harmonic constituents. Secondly, we ‘ see ’ each piece of information at a fixed point in space through a relatively narrow time window so that it would also appear hopeless to attempt to transform each of these relaxation time series into the Laplace domain where all of the previous discussion has been formulated. In the next section we see that these constraints oblige us to recast the discussion into a rather special form.

7. A pseudo-spectral form of the inverse theory

The foregoing brief remarks on the available Quaternary data set and the more comprehensive description in Peltier & Andrews (companion paper) has illustrated two distinct characteristics of these data. These characteristics force the theory to satisfy two important constraints: (a) since each gross earth datum refers to a specific (θ, ϕ) point on the Earth's surface so must the predictions of the model, and (b) since we ' see ' the response through a time window which varies in duration between each of the surface points at which we have data from times on the order of 5 K years to times on the order of 10 K years the predictions of the model must refer to fixed points in time. We must construct a point-by-point and time-by-time solution to the inverse problem. This would appear at the outset to be a much ' messier ' proposition to entertain than those which have been previously encountered in geophysical inverse problems. Fortunately the previous theoretical results allow us to accomplish this task in a straightforward fashion. The resulting computational structure is certainly no more difficult than that for the free oscillations problem to which the relaxation problem is, in a spectral sense, orthogonal. For the sake of explicitness we will confine our initial elaboration to the formalism for the inverse of one particular signature of the adjustment process: namely the absolute variations in local radius which are at least approximately frozen into the relative sea-level curves which have been thoroughly discussed in Peltier & Andrews (1976), but see the discussion in the second companion paper by Farrell & Clark (1976).

Suppose that the entire surface mass load $M(\theta, \phi)$ melted at a particular instant of time which we shall take to define the origin in time $t = 0$. From (20) we may show that the absolute change in local radius at a particular surface location (θ, ϕ) as a function of time t is given by

$$\Delta R(\theta, \phi, t) = \int d\Omega' \, G_R{}^H(\gamma, t) \, M(\theta', \phi') \tag{57}$$

where $d\Omega'$ is the element of surface area as before and where $G_R{}^H(\gamma, t)$ is the Heaviside Green function determined by substituting in equation (11) the Heaviside Love numbers (48) in place of their impulse response forms and where explicitly now

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') \tag{58}$$

is the cosine of the spherical angle separating source point (θ', ϕ') from field point (θ, ϕ) in the convolution integral (57). Since we assume that the mass functional M is known exactly we can decompose it spectrally into its various harmonic constituents as

$$M(\theta', \phi') = \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \mathcal{M}_{l'm'} \, Y_{l'm'}(\theta', \phi') \tag{59}$$

where $Y_{l'm'}(\theta', \phi')$ are the usual fully-normalized surface spherical harmonics (Jackson 1962). We also require the addition theorem for spherical harmonics (Jackson 1962) which states that

$$P_l(\cos\gamma) = \frac{4\Pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi') \, Y_{lm}(\theta, \phi) \tag{60}$$

where * indicates complex conjugation. In the definition of $G_R{}^H(\gamma, t)$ where

$$G_R{}^H(\gamma, t) = \frac{a}{m_e} \sum_{l=0}^{\infty} h_l{}^H(t) \, P_l(\cos\gamma) \tag{61}$$

we may substitute (60) and insert this spectral decomposition of $G_R{}^H(\gamma, t)$ into the solution of the direct problem (57) to obtain

$$\Delta R(\theta, \phi, t) = \int d\Omega' \frac{a}{m_e} \sum_{l=0}^{\infty} h_l^H(t)$$

$$\times \frac{4\Pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') \, Y_{lm}(\theta, \phi) \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \mathcal{M}_{l' m'} \, Y_{l' m'}(\theta', \phi'). \quad (62)$$

If we next operate through the summation signs in (62) with the surface integral and make use of the orthogonality relation between the $Y_{lm}(\theta, \phi)$, namely

$$\int d\Omega' \, Y_{lm}^*(\theta', \phi') \, Y_{l' m'}(\theta', \phi') = \delta_{ll'} \, \delta_{mm'} \quad (63)$$

then equation (62) is reduced to the form

$$\Delta R(\theta, \phi, t) = \frac{4\Pi a}{m_e} \sum_{l=0}^{\infty} \frac{h_l^H(t)}{2l+1} \sum_{m=-l}^{l} \mathcal{M}_{lm} \, Y_{lm}(\theta, \phi). \quad (64)$$

It is equation (64) which we shall employ in our solution of the inverse problem. It is pseudo-spectral in that no transformation of the observations (say $\Delta R(\theta, \phi, t)$) are required yet we assume that the mass functional M can be decomposed into its spherical harmonic constituents. Since we assume initially that M is exactly known this decomposition can be affected with arbitrary accuracy.

The first variation of equation (64) is then just

$$\delta \Delta R(\theta, \phi, t) = \frac{4\Pi a}{m_e} \sum_{l=0}^{\infty} \frac{\delta h_l^H(t)}{2l+1} \sum_{m=-l}^{l} \mathcal{M}_{lm} \, Y_{lm}(\theta, \phi). \quad (65)$$

Now $h_l^H(t)$ has been given previously in (50) (the viscous part) and its variation is just

$$\delta h_l^H(t) = \delta h_l^{H,V}(t) = \sum_j \frac{\partial h_l^{H,V}(t)}{\partial \bar{s}_j^l} \delta \bar{s}_j^l + \sum_j \frac{\partial h_l^{H,V}(t)}{\partial \bar{r}_j^l} \delta \bar{r}_j^l. \quad (66)$$

Where in (66) the partial derivatives are

$$\frac{\partial h_l^{H,V}(t)}{\partial \bar{s}_j^l} = \frac{\bar{r}_j^l}{\bar{s}_j^l} \left(t \, \bar{s}_j^l \exp(-\bar{s}_j^l t) - 1 + \exp(-\bar{s}_j^l t) \right) \quad (67)$$

$$\frac{\partial h_l^{H,V}(t)}{\partial \bar{r}_j^l} = \frac{(1 - \exp(-\bar{s}_j^l t))}{\bar{s}_j^l}. \quad (68)$$

Substitution of (68), (67) and (55) into (66) gives the simple form

$$\delta h_l^{H,V}(t) = \sum_j \frac{\bar{r}_j^l}{\bar{s}_j^l} \, t \exp(-\bar{s}_j^l t) \, \delta \bar{s}_j^l. \quad (69)$$

Inspection of (69) shows that this is exactly the result we would obtain by spectral convolution of (56) with a unit step followed by transformation to the time domain as must certaintly be the case. If we use (41) in (69) we obtain the following explicit expression for the first variation of $\delta h_l^{H,V}(t)$

$$\delta h_l^{H,V}(t) = \sum_j \bar{r}_j^l \, t \exp(-\bar{s}_j^l t) \frac{-\int_b^a dr \, r^2 K_l \frac{(\mu^2/\nu)}{(\bar{s}_j^l + (\mu/\nu))^2} \frac{\delta\nu}{\nu}}{\int_b^a dr \, r^2 K_l \frac{(\mu^2/\nu)}{(\bar{s}_j^l + (\mu/\nu))^2}}. \tag{70}$$

The integral in the denominator of (70) is of course just a normalization factor which ensures that the basic kernels are unimodular. Call this factor N_l^j. The integral in the numerator may be pulled outside the summation sign to give

$$\delta h_l^{H,V}(t) = \int_b^a dr \, r^2 \mathscr{K}_l(r,t) \frac{\delta\nu}{\nu} \tag{71}$$

where clearly

$$\mathscr{K}_l(r,t) = \sum_j \bar{r}_j^l \frac{t \exp(-\bar{s}_j^l t)}{N_l^j} K_l \frac{(\mu^2/\nu)}{(\bar{s}_j^l + (\mu/\nu))^2} \tag{72}$$

$$N_l^j = \int_b^a dr \, r^2 K_l \frac{(\mu^2/\nu)}{(\bar{s}_j^l + (\mu/\nu))^2}. \tag{73}$$

Next making use of (71) in (65) and again taking the integral outside the sum we obtain

$$\delta\Delta R(\theta,\phi,t) = \int_b^a dr \, r^2 \frac{\delta\nu}{\nu} \left[\frac{4\Pi a}{m_e} \sum_{l=0}^{\infty} \frac{\mathscr{K}_l(r,t)}{2l+1} \sum_{m=-l}^{l} \mathscr{M}_{lm} Y_{lm}(\theta,\phi) \right]. \tag{74}$$

The term in square brackets is clearly the Fréchet kernel (Backus & Gilbert 1967) for the space-time inverse. Equation (74) is one of the main results of this paper. With it we may study the inverse problem for mantle viscosity be assimilating directly the raised beach data which form the basic constituents of the Quaternary data set. This data will not have to be transformed in any way since equation (74) is an explicit relation between a radially distributed viscosity variation and the perturbation of the amplitude of the response at a particular (θ, ϕ) location on the surface and at a particular time t. We will be able to directly incorporate into the theory the real errors in the determination of the height above present sea level of a given beach and the real errors in the determination of the age of the beach via the C^{14} technique.

Exactly the same methods as those employed above may be used to construct appropriate kernels for the study of the gravity anomaly produced by the isostatic adjustment process and for the observed present-day rates of uplift. In fact, the kernel for present day rate of uplift (or at any time in the past) can be obtained from equation (74) simply by time differentiation. More specialized kernels may also be formed; for instance one may easily construct a kernel for the sensitivity of the non-tidal deceleration of the Earth's rotation to variations in the radial viscosity profile.

There is one minor point which remains to be settled regarding the previous development. This concerns the fact that we have assumed a very special form for the time history of the load, namely that it was entirely removed at one fixed instant. The required modification of theory necessary to incorporate a more general style of load removal is, however, rather straightforward and we shall give it here for completeness.

Any continuous history of load removal and application (redistribution) may be approximated by a series of discrete steps. We shall call such an approximation a model of ‘ stepwise discontinuous deglaciation ’. Such a model has the mathematical form

$$M(\theta, \phi, t) = \sum_{n=0}^{N} \mathscr{M}^n(\theta, \phi)\, H(t-t_n) \tag{75}$$

where the t_n are a series of discrete times bracketing the deglaciation history with $t_0 = 0$ being the assumed first instant of melting and $M^n(\theta, \phi)$ the space dependent load redistribution at time t_n. For $t < t_0$ the system is assumed to be in isostatic equilibrium. Returning to the forward problem as expressed in (20) we substitute from equation (75) to obtain

$$\Delta R(\theta, \phi, t) = \int d\Omega'\, G_R(\gamma, t-t') \sum_n \mathscr{M}^n(\theta, \phi)\, H(t'-t_n) \tag{76}$$

or equivalently in the Laplace transform domain

$$\Delta R(\theta, \phi, s) = \sum_n \int d\Omega' \frac{a}{m_e} \sum_{l=0}^{\infty} h_l(s)\, P_l(\cos\gamma)\, \mathscr{M}^n(\theta', \phi') \frac{\exp(-t_m s)}{s}. \tag{77}$$

Transformation of (77) back to the time domain gives

$$\Delta R(\theta, \phi, t) = \sum_n \int d\Omega' \frac{a}{m_e} \sum_{l=0}^{\infty} h_l^{H}(t-t_n)\, P_l(\cos\gamma)\, \mathscr{M}^n(\theta', \phi') \tag{78}$$

where

$$\begin{aligned} h_l^{H}(t-t_n) &= \sum_j \frac{\bar{r}_j^{\,l}}{\bar{s}_j^{\,l}}\, 1 - \exp\left[\left(-\bar{s}_j^{\,l}(t-t_n)\right)\right] + h_l^{E}\,\delta(t-t_n) \qquad \text{for } t \geqslant t_n \\ &= 0 \qquad \text{for } t < t_n. \end{aligned} \tag{79}$$

The differential kernels for the new problem involving the more complicated load removal history (75) are obtained in precisely the same way as before. The modification of the mathematical expression for the kernel is thus a simple one, involving a set of correctly phased (through t_n) and weighted (through $M^n(\theta', \phi')$) versions of the old kernel in (74).

8. Discussion

In the preceding sections we have demonstrated that the inverse theory for mantle viscosity is well posed within the framework of a first order perturbation theory and have derived the appropriate expressions for the differential kernels of the inhomogeneous problem. The only previous work on this subject of which I am aware is that due to Parsons (1972) whose discussion was confined to the half-space problem and dealt only with the homogeneous case. The theory given here is rigorous in the sense that it includes all of the essential physical ingredients of the adjustment process. This process is one which demands for its full understanding the treatment of the gravitational interaction between the time dependent surface mass load and the underlying planet within the correct topological framework. It is clearly not sufficient to consider the free decay characteristics of the medium and to base inferences of mantle viscosity upon them. We do not observe free decay times directly but rather the time dependent amplitude of some signature of the response to a specific applied load. It is the magnitude and surface distribution of the applied load which determines the final isostatically adjusted state.

In developing this formalism we have obtained a variational principle which applies to the free decay regime of a self-gravitating visco-elastic (Maxwell) spheroid.

In several respects this variational principle is similar to the one obtained earlier by Parsons (1972) for the non-gravitating half-space problem. Here however, it determines the change in position of any one of the multiplicity of normal modes of viscous gravitational relaxation belonging to a particular harmonic of order l. This principle is not restricted in its application to spheroids which have only radial perturbations of the radially stratified viscosity of their base state. It may also be applied in principle to determine corrections to the free decay times forced by lateral variations of viscosity. We can expect the normal modes to be split by such inhomogeneity. Again this problem has an exact analogue with determining the variations in eigenfrequency for the elastic gravitational free modes of oscillation which are forced by small lateral variations in elastic structure (Luh 1974). A further second-order correction which we might make is that due to the ellipticity of figure.

As stated in Paper 1, our objective in all of this is to attempt a rigorous examination of the extent to which a simple Maxwell model of the interior is capable of providing accord with the known Quaternary data set. We seek to understand, therefore, the extent to which a Newtonian model of the relaxation mechanism is an appropriate one in so far as isostatic adjustment is concerned. If it turns out that the Newtonian model is an appropriate one then by implication the mechanism by which mantle material ' creeps ' is essentially Herring–Nabarro (Herring 1950). In order to verify the validity of this implication we will have to demonstrate that alternative non-linear creep mechanisms are incapable of providing similar accord with the data. Whether or not this stage of the analysis can be successfully completed remains to be seen.

Initial experience with the forward problem (Cathles 1971, 1975; Peltier 1974; Peltier & Andrews 1976) has indicated that within the context of a spherical Maxwell model a uniform mantle viscosity of about 10^{22} P may be most appropriate.

McConnell (1965) and Peltier & Andrews (1976) respectively for the Fennoscandia and for the Laurentide data have shown that the presence of a lithosphere is an absolute necessity if one is to fit all the relaxation data. With an inviscid core this planetary model will be employed as a first guess in the inverse theory. If this model is reinforced by the calculation suggested here then the implications for work on mantle convection are important. It has been shown previously (Peltier 1972) that if the radial variation of mantle viscosity is not ' extreme ' then the radial mixing length for thermal convection in this system is liable to be on the order of the thickness of the mantle itself.

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Appendix A

Differential kernel for the shift in a free decay pole—a variational principle

In this appendix we deduce from the field equations (5) and (6) and the constitutive relation (2) the general variational principle stated in Section 5. We begin by reducing the constitutive relation (2) to a form which is more appropriate to the present application.

Define a compliance $K(s)$ such that (in analogy with the elastic problem)

$$K(s) = \lambda(s)+\tfrac{2}{3}\mu(s). \tag{80}$$

Direct substitution of the forms (3) and (4) for $\lambda(s)$ and $\mu(s)$ into (80) gives

$$K(s) \equiv K_e.$$

This follows from the simple physical fact that the Maxwell solid is viscously incompressible. Therefore the constitutive relation (2) may be rewritten as

$$\tilde{\tau}_{ij} = \left(K_e-\tfrac{2}{3}\mu(s)\right)\tilde{e}_{ii}\,\delta_{ij}+2\mu(s)\,\tilde{e}_{ij}. \tag{81}$$

Equation (81) may also be expanded in the following form (Malvern 1969):

$$\tilde{\tau}_{ij} = C_{ijkl}(s)\,\tilde{e}_{kl} \tag{82}$$

where the fourth rank tensor $C_{ijkl}(s)$ has the explicit form

$$C_{ijkl}(s) = \left(K_e-\tfrac{2}{3}\mu(s)\right)\delta_{ij}\,\delta_{kl}+\mu(s)\,(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk}) \tag{83}$$

where δ_{ij} is the usual Kroenecker delta function and the Einstein summation convention over repeated indices is assumed. Equation (23) follows because $C_{ijkl}(s)$ is assumed to be an isotropic tensor. A general discussion may be found in Malvern (1969).

We assume for our present purpose that the boundary conditions subject to which we are attempting to solve the field equations (5) and (6) are those corresponding to stress free conditions.

To determine the general form of the differential kernels for the free decay time history we operate on equation (5) by taking its inner product with a new solution vector $\tilde{v}_i$ which satisfies the same differential equations and boundary conditions as $\tilde{u}_i$. We similarly operate on equation (6) by multiplication with the perturbation in the gravitational potential $\tilde{\phi}^*$ which corresponds via Poisson's equation to the new displacement vector $\tilde{v}_i$. We divide the resulting equation from $(6)\times\tilde{\phi}^*$ by the factor

$(4\pi G)^{-1}$ add it to the equation which results from $(5)\times\tilde{v}_i$ and integrate over the infinite domain. The result is an energy relation which has the form

$$\int_E [\tilde{v}_i\partial_j\tilde{\tau}_{ij}-\tilde{v}_i\partial_i(\rho\tilde{u}_i g_i)-\rho\tilde{v}_i\partial_i\tilde{\phi}+\tilde{v}_i g_i\partial_j(\rho\tilde{u}_j)]\,dv$$
$$+\int_E [\tilde{\phi}^*\nabla^2\phi.(4\Pi G)^{-1}+\tilde{\phi}^*\partial_i(\rho\tilde{u}_i)]\,dv = 0. \tag{84}$$

In equation (24) E is the whole infinite space, $g_i=\partial_i\phi_0$ where ϕ_0 is the ambient gravitational potential which is assumed radial and the abbreviation $\partial_i=\partial/\partial x_i$.

We next want to reduce equation (84) to a more physically meaningful form. The first integral in (84) contains the displacement field (transformed) in every term so that this integral is equivalent to an integration over the Earth's volume since clearly $\tilde{u}_i$ vanishes outside. We next invoke Gauss theorem to reduce the first term in this integral using the tensor relation

$$\tilde{v}_i\partial_j\tilde{\tau}_{ij}=\partial_i(\tilde{v}_j.\tilde{\tau}_{ij})-(\partial_j\tilde{v}_i)\tilde{\tau}_{ij}. \tag{85}$$

By Gauss theorem the integral of the first term of (85) over the volume reduces to the integral of the normal component of $\tilde{v}_j\tilde{\tau}_{ij}$ over the surface. But the stress on the outer surface has been assumed to vanish thus this term gives zero identically. Thus under the integral sign in (84) we have

$$\tilde{v}_i\partial_j\tilde{\tau}_{ij}=-\partial_j\tilde{v}_i\tau_{ij}. \tag{86}$$

The first term in the second integral of (84) may be reduced using exactly the same procedure but here the bounding surface is taken at infinity where the perturbation in the potential field is assumed to vanish. Under the sign of the second integral in (84) we can then write

$$\tilde{\phi}^*\nabla^2\tilde{\phi}(4\Pi G)^{-1}=-\partial_i\tilde{\phi}^*.\partial_i\tilde{\phi}(4\Pi G)^{-1}. \tag{87}$$

Next we transfer the third term of the first integral in (84) into the second integral which is allowed since the domains over which the term is non-zero are the same. These operations reduce equation (84) to the slightly simpler form

$$\int_V [-\partial_j\tilde{v}_i\tilde{\tau}_{ij}-\tilde{v}_i\partial_i(\rho\tilde{u}_j g_j)+\tilde{v}_i g_i\partial_j(\rho\tilde{u}_j)]\,dv$$
$$+\int_E [-\partial_i\tilde{\phi}^*.\partial_i\tilde{\phi}(4\Pi G)^{-1}-\rho\tilde{v}_i\partial_i\tilde{\phi}-\rho\tilde{u}_i\partial_i\tilde{\phi}^*]\,dv=0. \tag{88}$$

The second term in the first integral may be written as

$$-\tilde{v}_i\partial_i(\rho\tilde{u}_j g_j)=-\tilde{v}_i[g_j\partial_i(\rho\tilde{u}_j)+\rho u_j\partial_i g_j]. \tag{89}$$

We can further reduce equation (88) by noting that

$$\begin{aligned}\partial_j\tilde{v}_i\tilde{\tau}_{ij}&=\partial_j\tilde{v}_i[C_{ijkl}(s)\tilde{e}_{kl}]\\&=K_e\partial_i\tilde{v}_i\partial_k\tilde{u}_k-\tfrac{2}{3}\mu(s)\partial_i\tilde{v}_i\partial_k\tilde{u}_k\\&\quad+\mu(s)\partial_j\tilde{v}_k\partial_j\tilde{u}_k+\mu(s)\partial_j\tilde{v}_k\partial_k\tilde{u}_j.\end{aligned} \tag{90}$$

But equation (90) has a very simple expression in terms of the Laplace transform of the strain deviator $\tilde{\Delta}_{ij}$ where

$$\tilde{\Delta}_{ij}=\tfrac{1}{2}(\partial_i\tilde{u}_j+\partial_j\tilde{u}_i)-\tfrac{1}{3}\partial_k\tilde{u}_k\delta_{ij} \tag{91}$$

and

$$\tilde{\Delta}^*_{ij}=\tfrac{1}{2}(\partial_i\tilde{v}_j+\partial_j\tilde{v}_i)-\tfrac{1}{3}\partial_k\tilde{v}_k\delta_{ij}$$

such that

$$\partial_j \tilde{v}_i \tilde{\tau}_{ij} = 2\tilde{\Delta}_{ij} \tilde{\Delta}_{ij}^* + K_e \partial_i \tilde{v}_i \partial_k \tilde{u}_k. \tag{92}$$

Inserting equations (92) and (89) into the Laplace transform domain energy relation (88) we have finally

$$\int_V [K_e \partial_i \tilde{v}_i \partial_k \tilde{u}_k + 2\mu(s) \tilde{\Delta}_{ij} \tilde{\Delta}_{ij}^* + \rho \tilde{v}_i^* \tilde{u}_j \partial_i \partial_j \phi_0 + \partial_j \phi_0 (\tilde{v}_i \partial_i \rho \tilde{u}_j - \tilde{v}_j \partial_i \rho \tilde{u}_i)] \, dv + \int_E dv [\partial_i \tilde{\phi}^* \partial_i \tilde{\phi} (4\Pi G)^{-1} + \rho \tilde{v}_i \partial_i \tilde{\phi} + \rho \tilde{u}_i \partial_i \tilde{\phi}^*] = 0. \tag{93}$$

It will be noted that this is precisely analogous to the equivalent elastic energy relation employed by Backus & Gilbert (1967) (their equation (29)), for application to the derivation of differential kernels for the frequencies of the normal modes of elastic-gravitational free oscillation of the Earth. By application of the correspondence principle we arrive at an energy relation which is precisely analogous to theirs but of course is valid in the Laplace transform domain of the imaginary frequency s rather than in the Fourier transform domain of the real frequency ω. To obtain the expression analogous to the Backus & Gilbert equation (29), with minor differences that are of no consequence, we simply take $\tilde{v}_i = \tilde{u}_i$ and $\tilde{\phi}^* = \tilde{\phi}$.

Next we employ the technique of perturbation operators to show that the Laplace transform domain energy relation (93) may be used to obtain a relationship between a small variation in the viscosity structure of the model $\delta\nu$ and the shift in a free decay pole (say any one of the $(-)\, \bar{s}_j^{\,l}$ introduced previously.

We may write our original field equations (5) and (6) in operator form as

$$\mathscr{L}\mathscr{S} = 0 \tag{94}$$

where $\mathscr{S} = (\tilde{u}_i, \tilde{\phi})$ is the solution 4-vector and $\mathscr{L}$ is a lumped differential operator, the form of which is obvious from the original field equations. In operator form we may thus represent the energy relation (93) in the form

$$\int_V \mathscr{S}^* . \mathscr{L} \mathscr{S} \, dv = 0 \tag{95}$$

where $\mathscr{S}^* = (\tilde{v}_i, \tilde{\phi}^*)$ is the new solution 4-vector introduced previously. But inspection of equation (93) shows immediately that

$$\int_V \mathscr{S}^* . \mathscr{L} \mathscr{S} \, dv = \int_V (\mathscr{L} \mathscr{S})^* . \mathscr{S} \, dv \tag{96}$$

so long as we are willing to restrict ourselves to consideration of real values of the Laplace transform variable s. Under this restriction $\mathscr{L}$ is a real differential operator and thus Hermitian. This important property of $\mathscr{L}$ for real s is the factor which makes it possible to obtain the general form of the variational principle which we are seeking.

We introduce a further abbreviation of notation by introducing the notion of an inner product on our space of 4 dimensional solutions $\mathscr{S}$ by defining

$$\langle \mathscr{S}^*, \mathscr{L} \mathscr{S} \rangle = \int_V \mathscr{S}^* . \mathscr{L} \mathscr{S} \, dv. \tag{97}$$

Then equation (96) may be simply represented as

$$\langle \mathscr{S}^*, \mathscr{L} \mathscr{S} \rangle = \langle \mathscr{L} \mathscr{S}^*, \mathscr{S} \rangle \tag{98}$$

where the operation of taking the complex conjugate denoted by *, does not affect $\mathscr{L}$ since it is a real operator as noted previously.

We next consider the functional ε defined by taking $\mathscr{S}^* = \mathscr{S}$ in the left-hand side of the energy relation (93) and enquiring as to the variation of this functional under small variations of the viscosity model $\delta\nu$.

If $\mathscr{S} = (\tilde{u}_i, \tilde{\phi})$ is a solution to (5) and (6) then by definition $\varepsilon \equiv 0$. Suppose that the rheological parameters are allowed to vary slightly such that $\nu \rightarrow \nu+\delta\nu$ keeping the density and the Lamé parameters fixed. Thus any free decay pole $s = s_j^{\,l}$ will shift slightly such that $s_j^{\,l} \rightarrow s_j^{\,l}+\delta s_j^{\,l}$ and the operator $\mathscr{L}$ since it depends both upon ν and upon s will become $\mathscr{L}+\delta\mathscr{L}$ in the process. For the new solution $\mathscr{S}+\delta\mathscr{S}$ and the new operator $\mathscr{L}+\delta\mathscr{L}$ we must have

$$\langle \mathscr{S}+\delta\mathscr{S}, (\mathscr{L}+\delta\mathscr{L})(\mathscr{S}+\delta\mathscr{S})\rangle = 0 \tag{99}$$

since (94) is a property of all solutions. For the sake of completeness we write out all terms of equation (99). It is equivalent to the eight terms

$$\begin{aligned}&\langle \mathscr{S}, \mathscr{L}\,\mathscr{S}\rangle + \langle \mathscr{S}, \mathscr{L}\,\delta\mathscr{S}\rangle + \langle \mathscr{S}, \delta\mathscr{L}\,\mathscr{S}\rangle + \langle \mathscr{S}, \delta\mathscr{L}\,\delta\mathscr{S}\rangle \\ &\quad + \langle \delta\mathscr{S}, \mathscr{L}\,\mathscr{S}\rangle + \langle \delta\mathscr{S}, \mathscr{L}\,\delta\mathscr{S}\rangle + \langle \delta\mathscr{S}, \delta\mathscr{L}\,\mathscr{S}\rangle + \langle \delta\mathscr{S}, \delta\mathscr{L}\,\delta\mathscr{S}\rangle = 0. \end{aligned} \tag{100}$$

We can reduce equation (100) as follows. First drop all terms of order δ^2 and higher because we are considering only small variations $\delta\nu$ in the viscosity model. This gives

$$\langle \mathscr{S}, \mathscr{L}\,\mathscr{S}\rangle + \langle \mathscr{S}, \mathscr{L}\,\delta\mathscr{S}\rangle + \langle \mathscr{S}, \delta\mathscr{L}\,\mathscr{S}\rangle + \langle \delta\mathscr{S}, \mathscr{L}\,\mathscr{S}\rangle = 0. \tag{101}$$

The first term in equation (101) is just $\varepsilon \equiv 0$ and the last term vanishes because $\mathscr{L}\,\mathscr{S} = 0$ is the original coupled set of differential equations. Furthermore the second term in equation (101) vanishes because of the previously stated Hermitian property of $\mathscr{L}$ for real values of s (equation (98)). Thus equation (101) reduces to

$$\langle \mathscr{S}, \delta\mathscr{L}\,\mathscr{S}\rangle = 0 \tag{102}$$

which is precisely the variational principle which we have been seeking.

It should be noticed that the structure of this variational principle is quite different from that due to Rayleigh and employed by Backus & Gilbert (1967) in their discussion of the elastic gravitational modes of free oscillation of the Earth, although it does have a superficial similarity. In that application the term $\langle \mathscr{S}, \mathscr{L}\delta\,\mathscr{S}\rangle$ vanishes because to first order the change in the eigenfunction $\mathscr{S}$, $\delta\mathscr{S}$, is *orthogonal* to $\mathscr{S}$ itself. Here the critical term vanishes because not only is $\mathscr{L}$ Hermitian but $\mathscr{L}\,\mathscr{S} \equiv 0$ since the deformation is assumed quasistatic (inertial forces are negligible). Furthermore the term $\langle \mathscr{S}, \mathscr{L}\,\mathscr{S}\rangle$ gives a finite contribution whereas here for the same reason as above, namely $\mathscr{L}\,\mathscr{S} = 0$, it vanishes identically.

To first-order equation (102) therefore states that the change in the solution vector $\mathscr{S}$ is unimportant, it is only the change in the s-dependent rheological parameters which must be taken into account. If we fix the elastic structure of the model, including the density since the Maxwell solid is viscously incompressible then since ε is stationary to first order, i.e. ε remains zero under small variations of rheology we must have from equation (93) that

$$\int_V dv[2\Delta_{ij}\,\Delta_{ij}^*\,\delta\mu(s)] = 0. \tag{103}$$

From equation (103) we can immediately deduce the expression for the differential kernel relating the shift in a free decay pole of the relaxation spectrum to a (small) change in the viscosity distribution, $\delta\nu$. This analysis is given in Section 5, where (103) is taken as the starting point.